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OTHER STRUCTURES OF RINGS OF INTEGERS NUMBERS, ISOMORPHIC BETWEEN THEM

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Abstract

As it is known, often solving a problem in a certain algebraic structure is quite difficult. That is why it is sometimes necessary to transfer the problem in an isomorphic structure with the given one and where it can be solved more easily. But the problem of determining isomorphic algebraic structures at once is quite difficult for pupils / students or even teachers. In this paper we aim to determine other isomorphic commutative ring structures with the ring of integers **Z**, on different subsets of the set **Z**, structures different from those known so far. For the beginning, we will see that if a and b are two integers, then on the set of integers at most equal to a, so on the set $\mathbf{Z}_{-\infty,a}$, and on the set of integers at least equal to b, so on the set $\mathbf{Z}_{b,+\infty}$, we will be able to define such a structure. On the other hand, as it is known, on the set of natural numbers **N** two internal operations can be defined so that they determine on **N** an isomorphic ring structure with **Z**. We will reach this result in this paper for the particular case b=0; moreover, such a structure of the ring we can also define on the set of non-positive integers, for the particular case a=0. In conclusion, we will show that, all known structures of rings of integers are isomorphic to each other and isomorphs with the ring (**Z**,+,.).

Keywords: Mathematics, Didactics of Mathematics, integer number, commutative ring, isomorphism

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1. Introduction

First, we specify that this paper is a continuation of the paper (Vălcan, 2020). So, everything I wrote there is also valid for this work. In addition, it should be noted that the topic proposed for study in this paper is a research topic in Didactics of Mathematics.

Research in Didactics of Mathematics, (which is nothing more than a search with specific reasoning and tools, to access a rational and / or spiritual knowledge in different directions), it is a relatively new field in the world, integrated in the sciences of education (Astolfi & Develey, 1989).

Today, research in the socio-human sciences (hence also in applied didactics), represents a transition from rational to spiritual knowledge; the current tendency being to unite the two ways of knowledge, (the interaction between facts and values takes place in the initial phases of scientific discovery, in the motivation of the approach or in "*inspiration*", and / or in the stage of connecting the new result to the scientific and social context, that is, to "*contemplate*" the result). The precise direction of research, the choice of methodologies, the coagulation of the results that entered the "*market of ideas*", the rapid application of conclusions in the daily process of learning Mathematics, all these are concerns of more recent date, fruit of collaboration between teachers, mathematicians, researchers, psychologists, etc. (Ausubel, 1968).

The requirements of a research in didactics must be formulated on the basis that, here, the "*object*" of research becomes the "*subject*" (he is man and society), and "*cause-effect*" relationships are replaced by "*involvement*" relationships, hence the need for interference between the two modes of access to knowledge (Vălcan 2013).

2. Problem Statement

As I mentioned in (Vălcan, 1997) in the research part, we consider the Didactics of Mathematics as an activity of mathematicians who are interested in learning Mathematics - as an educational discipline, using the methodologies and theories used in science, philosophy, psychology and / or pedagogy. From this point of view, at present, we can distinguish four major trends in Didactics of Mathematics:

(A) - the reconstruction of the contents of the Mathematics taught in the school, a reconstruction guided by an epistemological exigency;

(B) - designing, testing and evaluating new projects in mathematics education (not only Romanian), proposing in a coherent manner:

- purposes and objectives adapted to the current requirements of society,
- notional contents to keep up with the development of Mathematics as a science,
- activities and didactic means specific to learning Mathematics, which should take into account:

a) the cognitive processes through which the assimilation and understanding of students is achieved, and

b) the processes that appear in the communication relations of the teacher-social microgroup type.

(C) - intensive study of the acquisition of notions, formation of ideas and reasoning in students, for a better knowledge of their difficulties or possibilities, in learning Mathematics;

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(D) - the extensive evaluation of the competencies and attitudes developed in the students, by learning Mathematics, as well as the psycho-socio-economic implications of this learning.

Well, considering all this, we can say that the topic under study in this paper, as well as others, falls into Category (A).

3. Research Questions

In our research we will try to find answers to the following questions:

-There are others structures of commutative ring defined on sets of integers, apart from the known ones, and which are isomorphic to the commutative ring of integers, $(Z,+,\cdot)$?

-How can these structures be identified?

3.1. Regarding the first question

We are thinking here of sets of integers, unbounded inferior, but bordered superiorly, or vice versa.

3.2. Regarding the second question

We refer here to the ways of determining both these structures and the isomorphisms between them.

4. Purpose of the Study

Therefore, we answered the two questions in Paragraph 3. Thus, for any numbers a, $b\in Z$ there are two pairs of laws of internal composition on the sets $Z_{-\infty,a}$ and $Z_{b,+\infty}$, let's say „♣” and „♦”, respectively „♥” and „♠”, so that $(Z_{-\infty,a},♣,♦)$ and $(Z_{b,+\infty},♥,♠,)$ become commutative rings isomorphic to the ring $(Z,+,\cdot)$.

Concretely, on the set of integers at most equal to 3, which we denote with $\mathbf{Z}_{-\infty,3}$ and on the set of integers at least equal to 5, which we denote with $\mathbf{Z}_{5,+\infty}$, we can define two pairs of laws of internal composition so that let's say „♣” and „♦”, respectively „♥” and „♠”, so that $(\mathbf{Z}_{-\infty,3},♣,♦)$ and $(\mathbf{Z}_{5,+\infty},♥,♠)$ become commutative rings isomorphic to the ring $(\mathbf{Z},+,\cdot)$.

5. Research Methods

Let be a, $b\in\mathbf{Z}$. We note with:

$\mathbf{Z}_{-\infty,a}=\{x\in\mathbf{Z} \mid x\le a\}$ and $\mathbf{Z}_{b,+\infty}=\{x\in\mathbf{Z} \mid x\ge b\}$.

Then the functions:

$f_b : \mathbf{Z} \to \mathbf{Z}_{b,+\infty}$ and $g_a : \mathbf{Z} \to \mathbf{Z}_{-\infty,a}$

defined by:

$$f_b(x)=\begin{cases} b+2\cdot x & ,\text{if } x\ge 0 \\ b-2\cdot x-1, & \text{if } x<0 \end{cases} \quad \text{and} \quad g_a(x)=\begin{cases} a-2\cdot x & ,\text{if } x\ge 0 \\ a+2\cdot x+1, & \text{if } x<0 \end{cases}$$

are bijections, and their inverses are functions:

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$f_b^{-1} : \mathbf{Z}_{b,+\infty} \to \mathbf{Z}$ and $g_a^{-1} : \mathbf{Z}_{-\infty,a} \to \mathbf{Z}$,

defined by:

$$f_b^{-1}(x)=\begin{cases} \dfrac{x-b}{2}, & \text{if } x-b \text{ is even} \\ -\dfrac{(x-b)+1}{2}, & \text{if } x-b \text{ is odd} \end{cases} \quad \text{and} \quad g_a^{-1}(x)=\begin{cases} \dfrac{a-x}{2}, & \text{if } x-a \text{ is even} \\ -\dfrac{(a-x)+1}{2}, & \text{if } x-a \text{ is odd} \end{cases},$$

which, according to Vălcan, (2019) shows that:

$\mathbf{Z} \sim \mathbf{Z}_{b,+\infty}$ and $\mathbf{Z} \sim \mathbf{Z}_{-\infty,a}$,

whence it follows that:

$\mathbf{Z}_{b,+\infty} \sim \mathbf{Z}_{-\infty,a}$;

the bijection that accomplishes this is:

$h_{a,b}=g_a\circ f_b^{-1} : \mathbf{Z}_{b,+\infty} \to \mathbf{Z}_{-\infty,a}$;

where, for every $x\in\mathbf{Z}_{b,+\infty}$,

$$h_{a,b}(x)=g_a(f_b^{-1}(x))=\begin{cases} a-2\cdot f^{-1}(x), & \text{if } f^{-1}(x)\geq 0 \\ a+2\cdot f^{-1}(x)+1, & \text{if } f^{-1}(x)<0 \end{cases}=a+b-x$$

$$=a-(x-b),$$

and

$h_{a,b}^{-1}=f_b\circ g_a^{-1} : \mathbf{Z}_{-\infty,a} \to \mathbf{Z}_{b,+\infty}$;

where, for every $x\in\mathbf{Z}_{-\infty,a}$,

$$h_{a,b}^{-1}(x)=f_b(g_a^{-1}(x))=\begin{cases} b+2\cdot g^{-1}(x), & \text{if } g^{-1}(x)\geq 0 \\ b-2\cdot g^{-1}(x)-1, & \text{if } g^{-1}(x)<0 \end{cases}=a+b-x$$

$$=b+(a-x).$$

It follows that the following diagram (A) is commutative (see figure 1):

$$\begin{tikzcd} \mathbf{Z} \arrow[r, "f_b"] \arrow[d, "g_a"'] & \mathbf{Z}_{b,+\infty} \arrow[dl, "h_{a,b}=g_b\circ f_b^{-1}"] \\ \mathbf{Z}_{-\infty,a} & \end{tikzcd} \qquad \text{(A)}$$

Diagram (A)

The first fundamental result of this paragraph is:

Theorem 5.1**:** *For every number $a\in\mathbf{Z}$, there are two laws of internal composition, let's say „♣" and „♦", on the set $\mathbf{Z}_{-\infty,a}$, such that ($\mathbf{Z}_{-\infty,a}$,♣,♦) to become is a commutative ring isomorphic to the ring ($\mathbf{Z}$,+,·).*

Proof**:** We transfer the ring structure from $\mathbf{Z}$ to $\mathbf{Z}_{-\infty,a}$, using the function:

$g_a : \mathbf{Z} \to \mathbf{Z}_{-\infty,a}$,

where, for every $x\in\mathbf{Z}$,

$$g_a(x)=\begin{cases} a-2\cdot x &, \text{if } x\ge 0 \\ a+2\cdot x+1, & \text{if } x<0 \end{cases}$$

and

g_a^{-1} : $\mathbf{Z}_{-\infty,a} \to \mathbf{Z}$,

is defined by:

$$g_a^{-1}(x)=\begin{cases} \dfrac{a-x}{2} &, \text{if } a - x \text{ is even} \\ -\dfrac{(a-x)+1}{2} &, \text{if } a - x \text{ is odd} \end{cases}.$$

So, according to Vălcan (2017), we obtain the two laws of composition „♣” and „♦” on the set of integers $\mathbf{Z}_{-\infty,a}$. Let be x, y$\in\mathbf{Z}_{-\infty,a}$. For defining the law „♣”, we distinguish the following cases:

Case 1: a-x and a-y are even. Then:

$$x♣y=g(g^{-1}(x)+g^{-1}(y))=g\left(\frac{a-x}{2}+\frac{a-y}{2}\right)=g\left(\frac{2\cdot a-x-y}{2}\right)$$

$$=a-2\cdot\frac{2\cdot a-x-y}{2}=x+y-a$$

$$=a-(a-x)-(a-y).$$

Case 2: a-x and a-y are odd. Then:

$$x♣y=g(g^{-1}(x)+g^{-1}(y))=g\left(-\frac{(a-x)+1}{2}-\frac{(a-y)+1}{2}\right)$$

$$=g\left(\frac{-2\cdot a+x+y-2}{2}\right)=a+2\cdot\left(\frac{-2\cdot a+x+y-2}{2}\right)+1=x+y-a-1$$

$$=a-(a-x)-(a-y)-1.$$

Case 3: a-x is even and a-y is odd. Then:

$$x♣y=g(g^{-1}(x)+g^{-1}(y))=g\left(\frac{a-x}{2}-\frac{(a-y)+1}{2}\right)$$

$$=g\left(\frac{-x+y-1}{2}\right)=\begin{cases} a-(-x+y-1) &, \text{if } y\ge x+1 \\ a+(-x+y-1)+1, & \text{if } y<x+1 \end{cases}$$

$$=\begin{cases} a-(a-x)+(a-y)+1, & \text{if } y\ge x+1 \\ a+(a-x)-(a-y) &, \text{if } y<x+1 \end{cases}.$$

Case 4: a-x is odd and a-y is even. Then:

$$x♣y=g(g^{-1}(x)+g^{-1}(y))=g\left(\frac{-(a-x)-1}{2}+\frac{a-y}{2}\right)$$

$$=g\left(\frac{x-y-1}{2}\right)=\begin{cases} a-(x-y-1) &, \text{if } x\ge y+1 \\ a+(x-y-1)+1, & \text{if } x<y+1 \end{cases}$$

$$=\begin{cases} a+(a-x)-(a-y)+1, & \text{if } x\ge y+1 \\ a-(a-x)+(a-y) &, \text{if } x<y+1 \end{cases}.$$

Therefore, for every x, y$\in\mathbf{Z}_{-\infty,a}$.,

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$$x\clubsuit y=\begin{cases} a-(a-x)-(a-y) & \text{, if } a-x, a-y \text{ are even} \\ a-(a-x)-(a-y)-1 & \text{, if } a-x, a-y \text{ are odd,} \\ a-(a-x)+(a-y)+1 & \text{, if } a-x \text{ is even, } a-y \text{ is odd and } y \geq x+1 \\ a+(a-x)-(a-y) & \text{, if } a-x \text{ is even, } a-y \text{ is odd and } y < x+1 \\ a+(a-x)-(a-y)+1 & \text{, if } a-x \text{ is odd, } a-y \text{ is even and } x \geq y+1 \\ a-(a-x)+(a-y) & \text{, if } a-x \text{ is odd, } a-y \text{ is even and } x < y+1 \end{cases}.$$

Now, for defining the law „♦”, we distinguish the following cases:

Case 1: a-x and a-y are even. Then:

$$x\blacklozenge y=g(g^{-1}(x)\cdot g^{-1}(y))=g\left(\frac{a-x}{2}\cdot\frac{a-y}{2}\right)=g\left(\frac{(a-x)\cdot(a-y)}{4}\right)=a-2\cdot\frac{(a-x)\cdot(a-y)}{4}$$

$$=a-\frac{(a-x)\cdot(a-y)}{2}.$$

Case 2: a-x and a-y are odd. Then:

$$x\blacklozenge y=g(g^{-1}(x)\cdot g^{-1}(y))=g\left(\frac{(a-x)+1}{2}\cdot\frac{(a-y)+1}{2}\right)=a-2\cdot\frac{[(a-x)+1]}{2}\cdot\frac{[(a-y)+1]}{2}$$

$$=a-\frac{[(a-x)+1]\cdot[(a-y)+1]}{2}.$$

Case 3: a-x is even and a-y is odd. Then:

$$x\blacklozenge y=g(g^{-1}(x)\cdot g^{-1}(y))=g\left(-\frac{a-x}{2}\cdot\frac{(a-y)+1}{2}\right)$$

$$=a-\frac{(a-x)\cdot[(a-y)+1]}{2}+1.$$

Case 4: a-x is odd and a-y is even. Then:

$$x\blacklozenge y=g(g^{-1}(x)\cdot g^{-1}(y))=g\left(\frac{-(a-x)-1}{2}\cdot\frac{a-y}{2}\right)$$

$$=a-\frac{(a-y)\cdot[(a-x)+1]}{2}+1.$$

Therefore, for every x, $y\in\mathbf{Z}_{-\infty,a}$,

$$x\blacklozenge y=\begin{cases} a-\dfrac{(a-x)\cdot(a-y)}{2} & \text{, if } a-x, a-y \text{ are even} \\ a-\dfrac{[(a-x)+1]\cdot[(a-y)+1]}{2} & \text{, if } a-x, a-y \text{ are odd,} \\ a-\dfrac{(a-x)\cdot[(a-y)+1]}{2}+1 & \text{, if } a-x \text{ is even and } a-y \text{ is odd,} \\ a-\dfrac{[(a-x)+1]\cdot(a-y)}{2}+1 & \text{, if } a-x \text{ is odd and } a-y \text{ is even,} \end{cases}.$$

On the other hand,

$$e_{Z_{-\infty,a}}=g_a(e_Z)=g_a(0)=a$$

and

$$-x_{Z_{-\infty,a}}=g_a(-g_a^{-1}(x))=\begin{cases} a-(a-x)-1, & \text{if } a-x \text{ is odd} \\ a-(a-x)+1, & \text{if } a-x \text{ is even} \end{cases}$$

$$=\begin{cases} x-1, \text{ if } a-x \text{ is odd} \\ x+1, \text{ if } a-x \text{ is even} \end{cases},$$

and:

$1_{Z_{-\infty,a}}=g_a(1)=a-2$

and

$$x^{-1}_{Z_{-\infty,a}}=g_a\left(\frac{1}{g_a^{-1}(x)}\right)=\begin{cases} g_a\left(\dfrac{2}{a-x}\right) & \text{, if } a-x \text{ is even} \\ g_a\left(-\dfrac{2}{a-x+1}\right) & \text{, if } a-x \text{ is odd} \end{cases}$$

$$=\begin{cases} a-\dfrac{4}{a-x} & \text{, if } a-x \text{ is even} \\ a-\dfrac{4}{a-x+1}+1 & \text{, if } a-x \text{ is odd} \end{cases}.$$

But, if a-x is even, then:

$\dfrac{2}{a-x}\in\mathbf{Z}$ if and only if $a-x=2$,

that is:

$x=a-2=g_a(1)$,

and if a-x is odd, then:

$\dfrac{2}{a-x+1}\in\mathbf{Z}$ if and only if $a-x=1$,

that is:

$x=a-1=g_a(-1)$.

Therefore, according to Vălcan (2017), $(\mathbf{Z}_{-\infty,a},\clubsuit,\blacklozenge)$ is a commutative ring isomorphic to the ring $(\mathbf{Z},+,\cdot)$, and the only invertible elements in the ring $\mathbf{Z}_{-\infty,a}$ are:

a-2 and a-1.

Now let's show that, indeed, the function:

$g_a : \mathbf{Z} \to \mathbf{Z}_{-\infty,a}$,

defined by: for any $x\in\mathbf{Z}$,

$$g_a(x)=\begin{cases} a-2\cdot x & \text{, if } x\geq 0 \\ a+2\cdot x+1 & \text{, if } x<0 \end{cases},$$

is an isomorphism between the two rings. For this, we first notice that, for every x, $y\in\mathbf{Z}$,

$$g_a(x+y)=\begin{cases} a-2\cdot(x+y) & \text{, if } x+y\geq 0 \\ a+2\cdot(x+y)+1 & \text{, if } x+y<0 \end{cases}.$$

Then:

$$a-g_a(x)=\begin{cases} 2\cdot x & \text{, if } x\geq 0 \\ -2\cdot x-1 & \text{, if } x<0 \end{cases} \quad \text{and} \quad a-g_a(y)=\begin{cases} 2\cdot y & \text{, if } y\geq 0 \\ -2\cdot y-1 & \text{, if } y<0 \end{cases}.$$

To determine the expression $g_a(x)\clubsuit g_a(y)$ we distinguish the following cases:

Case 1: $a-g_a(x)$ and $a-g_a(y)$ are even. Then:

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$g_a(x)♣g_a(y)=a-(a-g_a(x))-(a-g_a(y))=g_a(x)+g_a(y)-a$

$=a-2\cdot(x+y)$.

Case 2: $a-g_a(x)$ and $a-g_a(y)$ are odd. Then:

$g_a(x)♣g_a(y)=a-(a-g_a(x))-(a-g_a(y))-1=g_a(x)+g_a(y)-a-1$

$=a+2\cdot(x+y)+1$.

Case 3: $a-g_a(x)$ is even and $a-g_a(y)$ is odd. Then:

$$g_a(x)♣g_a(y)=\begin{cases} a-(a-g_a(x))+(a-g_a(y))+1, \text{if } g_a(y)\geq g_a(x)+1 \\ a+(a-g_a(x))-(a-g_a(y)) \quad , \text{if } g_a(y)< g_a(x)+1 \end{cases}$$

$$=\begin{cases} a-2\cdot(x+y) \quad , \text{if } x+y\geq 0 \\ a+2\cdot(x+y)+1, \text{if } x+y<0 \end{cases}.$$

Case 4: $a-g_a(x)$ is odd and $a-g_a(y)$ is even. Then:

$$g_a(x)♣g_a(y)=\begin{cases} a+(a-g_a(x))-(a-g_a(y))+1, \text{if } g_a(x)\geq g_a(y)+1 \\ a-(a-g_a(x))+(a-g_a(y)) \quad , \text{if } g_a(x)< g_a(y)+1 \end{cases}$$

$$=\begin{cases} a-2\cdot(x+y) \quad , \text{if } x+y\geq 0 \\ a+2\cdot(x+y)+1, \text{if } x+y<0 \end{cases}.$$

It follows that for every $x, y\in\mathbf{Z}$:

$g_a(x+y)=g_a(x)♣g_a(y)$.

Now, to determine the expression $g_a(x)♦g_a(y)$ we distinguish the following cases:

Case 1: $a-g_a(x)$ and $a-g_a(y)$ are even. Then $x\geq0$ and $y\geq0$, and

$$g_a(x)♦g_a(y)=a-\frac{(a-g_a(x))\cdot(a-g_a(y))}{2}$$

$=a-2\cdot x\cdot y$.

Case 2: $a-g_a(x)$ and $a-g_a(y)$ are odd. Then $x<0$ and $y<0$, and:

$$g_a(x)♦g_a(y)=a-\frac{[(a-g_a(x))+1]\cdot[(a-g_a(y))+1]}{2}$$

$=a-2\cdot x\cdot y$.

Case 3: $a-g_a(x)$ is even and $a-g_a(y)$ is odd. Then $x\geq0$ and $y<0$, and:

$$g_a(x)♦g_a(y)=a-\frac{(a-g_a(x))\cdot[(a-g_a(y))+1]}{2}+1$$

$=a+2\cdot x\cdot y+1$.

Case 4: $a-g_a(x)$ is odd and $a-g_a(y)$ is even. Then $x<0$ and $y\geq0$, and:

$$g_a(x)♦g_a(y)=a-\frac{(a-g_a(y))\cdot[(a-g_a(x))+1]}{2}+1$$

$=a+2\cdot x\cdot y+1$.

On the other hand,

$$g_a(x\cdot y)=\begin{cases} a-2\cdot(x\cdot y) \quad , \text{if } x\cdot y\geq 0 \\ a+2\cdot(x\cdot y)+1, \text{if } x\cdot y<0 \end{cases}.$$

So, for every $x, y\in\mathbf{Z}$,

$g_a(x\cdot y)=g_a(x)♦g_a(y)$.

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Therefore, according to Vălcan (2017), ($\mathbf{Z}_{-\infty,a}$,♣,♦) is a commutative ring isomorphic to the ring ($\mathbf{Z}$,+,·).

Remark 5.2: *For a=0, from Theorem 5.1, obtain the ring structure on the set $\mathbf{Z}_{-\infty,0}$, of non-positive integers, transferred from the ring ($\mathbf{Z}$,+,·) by function*:

$g_0 : \mathbf{Z}_{-\infty,0} \to \mathbf{Z}$,

defined by:

$$g_0(x)=\begin{cases} -2\cdot x \quad ,if\ x\geq 0 \\ 2\cdot x+1, if\ x<0 \end{cases}$$

and for which the inverse is:

$g_0^{-1} : \mathbf{Z}_{-\infty,0} \to \mathbf{Z}$,

defined by:

$$g_0^{-1}(x)=\begin{cases} -\dfrac{x}{2} \ ,if\ x \text{ is even} \\ \dfrac{x-1}{2}, if\ x \text{ is odd} \end{cases}.$$

The second fundamental result of this paper is:

Theorem 5.3: *For every b∈$\mathbf{Z}$, there are two laws of internal composition, let's say „♥" and „♠", on the set $\mathbf{Z}_{b,+\infty}$, such that ($\mathbf{Z}_{b,+\infty}$, ♥,♠) to become is a commutative ring isomorphic to the ring ($\mathbf{Z}$,+,·).*

Proof: We transfer the ring structure from $\mathbf{Z}$ to $\mathbf{Z}_{b,+\infty}$, using the bijection function:

$f_b : \mathbf{Z} \to \mathbf{Z}_{b,+\infty}$,

where, for every x∈$\mathbf{Z}$,

$$f_b(x)=\begin{cases} b+2\cdot x \quad\ , if\ x\geq 0 \\ b-2\cdot x-1, if\ x<0 \end{cases},$$

and

$f_b^{-1} : \mathbf{Z}_{b,+\infty} \to \mathbf{Z}$,

is defined by:

$$f_b^{-1}(x)=\begin{cases} \dfrac{x-b}{2} \quad\quad , if\ x-b \text{ is even} \\ -\dfrac{(x-b)+1}{2}, if\ x-b \text{ is odd} \end{cases}.$$

Hence, according to Vălcan (2017), obtain the two composition laws „♥" and „♠" on $\mathbf{Z}_{b,+\infty}$. Let be x, y∈$\mathbf{Z}_{b,+\infty}$. For defining the law „♥", we distinguish the following cases:

<u>Case 1</u>: x-b and y-b are even. Then:

$$x♥y=f_b(f_b^{-1}(x)+f_b^{-1}(y))=f_b\left(\frac{x-b}{2}+\frac{y-b}{2}\right)=f_b\left(\frac{x+y-2\cdot b}{2}\right)$$

$$=b+2\cdot\frac{x+y-2\cdot b}{2}=x+y-b$$

$$=b+(x-b)+(y-b).$$

<u>Case 2</u>: x-b and y-b are odd. Then:

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$$x♥y=f_b(f_b^{-1}(x)+f_b^{-1}(y))=f_b\left(-\frac{(x-b)+1}{2}-\frac{(y-b)+1}{2}\right)$$

$$=f_b\left(\frac{2\cdot b-x-y-2}{2}\right)=b\text{-}2\cdot\left(\frac{2\cdot b-x-y-2}{2}\right)\text{-}1=x+y\text{-}b+1$$

$$=b+(x\text{-}b)+(y\text{-}b)+1.$$

Case 3: x-b is even and y-b is odd. Then:

$$x♥y=f_b(f_b^{-1}(x)+f_b^{-1}(y))=f_b\left(\frac{x-b}{2}-\frac{(y-b)+1}{2}\right)$$

$$=f_b\left(\frac{x-y-1}{2}\right)=\begin{cases}b+(x-y-1) & ,\text{if } x\geq y+1\\ b-(x-y-1)-1, & \text{if } x<y+1\end{cases}$$

$$=\begin{cases}b+(x-b)-(y-b)-1, & \text{if } x\geq y+1\\ b-(x-b)+(y-b) & ,\text{if } x<y+1\end{cases}.$$

Case 4: x-b is odd and y-b is even. Then:

$$x♥y=f_b(f_b^{-1}(x)+f_b^{-1}(y))=f_b\left(\frac{-(x-b)-1}{2}+\frac{y-b}{2}\right)$$

$$=f_b\left(\frac{-x+y-1}{2}\right)=\begin{cases}b+(-x+y-1) & ,\text{if } y\geq x+1\\ b-(-x+y-1)+1, & \text{if } y<x+1\end{cases}$$

$$=\begin{cases}b-(x-b)+(y-b)-1, & \text{if } y\geq x+1\\ b+(x-b)-(y-b) & ,\text{if } y<x+1\end{cases}.$$

Therefore, for every x, $y\in\mathbf{Z}_{b,+\infty}$,

$$x♥y=\begin{cases}b+(x-b)+(y-b) & ,\text{if } x\text{-}b \text{ and } y\text{-}b \text{ are even}\\ b+(x-b)+(y-b)+1, & \text{if } x\text{-}b \text{ and } y\text{-}b \text{ are odd,}\\ b+(x\text{-}b)\text{-}(y\text{-}b)-1 & ,\text{if } x\text{-}b \text{ is even, } y\text{-}b \text{ is odd and } x\geq y+1\\ b\text{-}(x\text{-}b)+(y\text{-}b) & ,\text{if } x\text{-}b \text{ is even, } y\text{-}b \text{ is odd and } x<y+1\\ b\text{-}(x\text{-}b)+(y\text{-}b)\text{-}1 & ,\text{if } x\text{-}b \text{ is odd, } y\text{-}b \text{ is even and } y\geq x+1\\ b+(x\text{-}b)\text{-}(y\text{-}b) & ,\text{if } x\text{-}b \text{ is odd, } y\text{-}b \text{ is even and } y<x+1\end{cases}.$$

Now, defining the law „♠”, we distinguish the following cases:

Case 1: x-b and y-b are even. Then:

$$x♠y=f_b(f_b^{-1}(x)\cdot f_b^{-1}(y))=f_b\left(\frac{x-b}{2}\cdot\frac{y-b}{2}\right)=f_b\left(\frac{(x-b)\cdot(y-b)}{4}\right)=b+2\cdot\frac{(x-b)\cdot(y-b)}{4}$$

$$=b+\frac{(x-b)\cdot(y-b)}{2}.$$

Case 2: x-b and y-b are odd. Then:

$$x♠y=f_b(f_b^{-1}(x)\cdot f_b^{-1}(y))=f_b\left(\frac{(x-b)+1}{2}\cdot\frac{(y-b)+1}{2}\right)=b+2\cdot\frac{[(x-b)+1]}{2}\cdot\frac{[(y-b)+1]}{2}$$

$$=b+\frac{[(x-b)+1]\cdot[(y-b)+1]}{2}.$$

Case 3: x-b is even and y-b is odd. Then:

$$x♠y=f_b(f_b^{-1}(x)\cdot f_b^{-1}(y))=f_b\left(-\frac{x-b}{2}\cdot\frac{(y-b)+1}{2}\right)$$

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$=b+\frac{(x-b)\cdot[(y-b)+1]}{2}-1.$

Case 4: x-b is odd and y-b is even. Then:

$x♠y=f_b(f_b^{-1}(x)\cdot f_b^{-1}(y))=f_b\left(\frac{-(x-b)-1}{2}\cdot\frac{y-b}{2}\right)$

$=b+\frac{(y-b)\cdot[(x-b)+1]}{2}-1.$

Therefore, for every x, $y\in\mathbf{Z}_{b,+\infty}$,

$$x♠y=\begin{cases} b+\frac{(x-b)\cdot(y-b)}{2} & \text{, if } x\text{ - }b \text{ and } y\text{ - }b \text{ are even} \\ b+\frac{[(x-b)+1]\cdot[(y-b)+1]}{2} & \text{, if } x\text{ - }b \text{ and } y\text{ - }b \text{ are odd,} \\ b+\frac{(x-b)\cdot[(y-b)+1]}{2}-1 & \text{, if } x\text{ - }b \text{ is even and } y\text{ - }b \text{ is odd,} \\ b+\frac{[(x-b)+1]\cdot(y-b)}{2}-1 & \text{, if } x\text{ - }b \text{ is odd and } y\text{ - }b \text{ is even,} \end{cases}.$$

On the other hand,

$e_{Z_{b,+\infty}}=f_b(e_Z)=f_b(0)=b$

and

$$-x_{Z_{b,+\infty}}=f_b(-f_b^{-1}(x))=\begin{cases} b+(x-b)-1, & \text{if } x\text{ - }b \text{ is even and } x\text{ - }b\geq 2 \\ b & \text{, if } x=b \\ b+(x-b)+1, & \text{if } x\text{ - }b \text{ is odd} \end{cases}$$

$$=\begin{cases} x-1, & \text{if } x\text{ - }b \text{ is even and } x\text{ - }b\geq 2 \\ b & \text{, if } x=b \\ x+1, & \text{if } x\text{ - }b \text{ is odd} \end{cases},$$

and:

$1_{Z_{b,+\infty}}=f_b(1)=b+2$

and

$$x^{-1}_{Z_{a,+\infty}}=f_b\left(\frac{1}{f_b^{-1}(x)}\right)=\begin{cases} f_b\left(\frac{2}{x-b}\right) & \text{, if } x\text{ - }b \text{ is even} \\ f_b\left(-\frac{2}{x-b+1}\right), & \text{if } x\text{ - }b \text{ is odd} \end{cases}$$

$$=\begin{cases} b+\frac{4}{x-b} & \text{, if } x\text{ - }b \text{ is even} \\ b+\frac{4}{x-b+1}-1, & \text{if } y\text{ - }b \text{ is odd} \end{cases}.$$

But, if x-b is even, then:

$\frac{2}{x-b}\in\mathbf{Z}$ if and only if x-b=2,

that is:

$x=b+2=f_b(1)$,

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and if x-b is odd, then:

$\frac{2}{x-b+1} \in \mathbf{Z}$ if and only if x-b=1,

that is:

$x=b+1=f_b(-1)$.

Therefore, according to Vălcan (2017), ($\mathbf{Z}_{b,+\infty}$,♥,♠) is a commutative ring isomorphic to the ring ($\mathbf{Z}$,+,·), and the only invertible elements in the ring $\mathbf{Z}_{b,+\infty}$ are:

b+2 and b+1.

Now let's show that, indeed, the function:

$f_b : \mathbf{Z} \to \mathbf{Z}_{b,+\infty}$,

defined by: for every $x \in \mathbf{Z}$,

$$f_b(x)=\begin{cases} b+2\cdot x & ,\text{if } x\geq 0 \\ b-2\cdot x-1, & \text{if } x<0 \end{cases}$$

is an isomorphism between the two rings. For every x, $y \in \mathbf{Z}$,

$$f_b(x+y)=\begin{cases} b+2\cdot(x+y) & ,\text{if } x+y\geq 0 \\ b-2\cdot(x+y)-1, & \text{if } x+y<0 \end{cases}.$$

Then:

$$f_b(x)-b=\begin{cases} 2\cdot x & ,\text{if } x\geq 0 \\ -2\cdot x-1, & \text{if } x<0 \end{cases} \quad \text{and} \quad f_b(y)-b=\begin{cases} 2\cdot y & ,\text{if } y\geq 0 \\ -2\cdot y-1, & \text{if } y<0 \end{cases}.$$

Now we determine the value of $f_b(x)$♥$f_b(y)$. We distinguish the following cases:

Case 1: $f_b(x)$-b and $f_b(y)$-b are even. Then x≥0 and y≥0, and:

$f_b(x)$♥$f_b(y)=b+(f_b(x)-b)+(f_b(y)-b)=f_b(x)+f_b(y)-b=b+2\cdot(x+y)$.

Case 2: $f_b(x)$-b and $f_b(y)$-b are odd. Then x<0 and y<0, and:

$f_b(x)$♥$f_b(y)=b+(f_b(x)-b)+(f_b(y)-b)+1=f_b(x)+f_b(y)-b+1=b-2\cdot(x+y)-1$.

Case 3: $f_b(x)$-b is even and $f_b(y)$-b is odd. Then x≥0 and y<0, and:

$$f_b(x)♥f_b(y)=\begin{cases} b+(f_b(x)-b)-(f_b(y)-b)-1, & \text{if } f_b(x)\geq f_b(y)+1 \\ b-(f_b(x)-b)+(f_b(y)-b) & ,\text{if } f_b(x)<f_b(y)+1 \end{cases}$$

$$=\begin{cases} b+2\cdot(x+y) & ,\text{if } x+y\geq 0 \\ b-2\cdot(x+y)-1, & \text{if } x+y<0 \end{cases}.$$

Case 4: $f_b(x)$-b is odd and $f_b(y)$-b is even. Then:

$$f_b(x)♥f_b(y)=\begin{cases} b-(f_b(x)-b)+(f_b(y)-b)-1, & \text{if } f_b(x)\geq f_b(y)+1 \\ b+(f_b(x)-b)-(f_b(y)-b) & ,\text{if } f_b(x)<f_b(y)+1 \end{cases}$$

$$=\begin{cases} b+2\cdot(x+y) & ,\text{if } x+y\geq 0 \\ b-2\cdot(x+y)-1, & \text{if } x+y<0 \end{cases}.$$

It foolows that, for every x, $y \in \mathbf{Z}$:

$f_b(x+y)=f_b(x)$♥$f_b(y)$.

Now, to determine the expression $f_b(x)$♠$f_b(y)$, we distinguish the following cases:

Case 1: $f_b(x)$-b and $f_b(y)$-b are even. Then, x≥0 and y≥0, and:

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$f_b(x)♠f_b(y)=b+\frac{(f_b(x)-b)\cdot(f_b(y)-b)}{2}=b+2\cdot x\cdot y$.

<u>Case 2</u>: $f_b(x)-b$ and $f_b(y)-b$ are odd. Then, $x<0$ and $y<0$, and:

$f_b(x)♠f_b(y)=b+\frac{[(f_b(x)-b)+1]\cdot[(f_b(y)-b)+1]}{2}=b+2\cdot x\cdot y$.

<u>Case 3</u>: $f_b(x)-b$ is even and $f_b(y)-b$ is odd. Then, $x\geq 0$ and $y<0$, and:

$f_b(x)♠f_b(y)=b+\frac{(f_b(x)-b)\cdot[(f_b(y)-b)+1]}{2}-1=f-2\cdot x\cdot y-1$.

<u>Case 4</u>: $f_b(x)-b$ is odd and $f_b(y)-b$ is even. Then:

$f_b(x)♠f_b(y)=b+\frac{(f_b(y)-b)\cdot[(f_b(x)-b)+1]}{2}-1=b-2\cdot x\cdot y-1$.

On the other hand,

$f_b(x\cdot y)=\begin{cases} b+2\cdot(x\cdot y) & ,\text{if } x\cdot y\geq 0 \\ b-2\cdot(x\cdot y)-1, & \text{if } x\cdot y<0 \end{cases}$.

Hence, for every $x, y\in\mathbf{Z}$,

$f_b(x\cdot y)=f_b(x)♠f_b(y)$.

Now, we can say that the theorem is completely proved.

At the end of this paragraph, three further remarks are required:

Remark 5.4: *For $b=0$ we get the structure of commutative ring on the set $\mathbf{Z}_{-\infty,0}=\mathbf{N}$, of non-negative integers, transferred from the ring $(\mathbf{Z},+,\cdot)$ by function*:

$f_0:\mathbf{Z}\to\mathbf{Z}_{0,+\infty}$,

defined by:

$f_0(x)=\begin{cases} 2\cdot x & ,\text{if } x\geq 0 \\ -2\cdot x-1, & \text{if } x<0 \end{cases}$,

whose inverse is the function:

$f_0^{-1}:\mathbf{Z}_{0,+\infty}\to\mathbf{Z}$,

defined by:

$f_0^{-1}(x)=\begin{cases} \frac{x}{2} & ,\text{if } x \text{ is even} \\ -\frac{x+1}{2}, & \text{if } x \text{ is odd} \end{cases}$.

Remark 5.5: *As demonstrated above, for any a, $b\in\mathbf{Z}$, the rings $(\mathbf{Z}_{-\infty,a},♣,♦)$ and $(\mathbf{Z}_{b,+\infty},♥,♠)$ are commutative, and the diagram (A) is a commutative diagram of commutative rings*.

Remark 5.6: *For every number $a\in\mathbf{Z}$, there are two laws of internal composition, let's say „♣" and „♦", on the set $\mathbf{Z}_{-\infty,a}$, such that $(\mathbf{Z}_{-\infty,a},♣,♦)$ to become is a commutative ring isomorphic to the ring $(\mathbf{Z},+,\cdot)$ and there are two laws of internal composition, let's say „♥" and „♠", on the set $\mathbf{Z}_{a,+\infty}$, such that $(\mathbf{Z}_{a,+\infty},♥,♠)$ to become is a commutative ring isomorphic to the ring $(\mathbf{Z},+,\cdot)$ and so that the following diagram (B) is a commutative diagram of commutative rings*:

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$$\begin{tikzcd} \mathbf{Z} \arrow[r, "f_a"] \arrow[d, "g_a"'] & \mathbf{Z}_{a,+\infty} \arrow[dl, "h_a=g_a\circ f_a^{-1}"] \\ \mathbf{Z}_{-\infty,a} & \end{tikzcd} \quad . \qquad \text{(B)}$$

Diagram (B)

In the diagram (B) (see figure 2);

$\mathbf{Z}_{a,+\infty}=\{x\in\mathbf{Z} \mid x\geq a\}$ and $\mathbf{Z}_{-\infty,a}=\{x\in\mathbf{Z} \mid x\leq a\}$.

and functions:

$f_a : \mathbf{Z} \to \mathbf{Z}_{a,+\infty}$ and $g_a : \mathbf{Z} \to \mathbf{Z}_{-\infty,a}$

defined by:

$$f_a(x)=\begin{cases} a+2\cdot x & , \text{if } x\geq 0 \\ a-2\cdot x-1, & \text{if } x<0 \end{cases} \quad \text{and} \quad g_a(x)=\begin{cases} a-2\cdot x & , \text{if } x\geq 0 \\ a+2\cdot x+1, & \text{if } x<0 \end{cases},$$

are bijections (they are precisely isomorphisms of rings), whose inverses are:

$f_a^{-1} : \mathbf{Z}_{a,+\infty} \to \mathbf{Z}$ and $g_a^{-1} : \mathbf{Z}_{-\infty,a} \to \mathbf{Z}$,

defined by:

$$f_a^{-1}(x)=\begin{cases} \dfrac{x-a}{2} & , \text{if } x-a \text{ is even} \\ -\dfrac{(x-a)+1}{2}, & \text{if } x-a \text{ is odd} \end{cases} \quad \text{and} \quad g_a^{-1}(x)=\begin{cases} \dfrac{a-x}{2} & , \text{if } x-a \text{ is even} \\ -\dfrac{(a-x)+1}{2}, & \text{if } x-a \text{ is odd} \end{cases}.$$

The function that achieves the isomorphism between the rings ($\mathbf{Z}_{-\infty,a}$,♣,♦) și ($\mathbf{Z}_{a,+\infty}$,♥,♠) is:

$h_{a,a}=g_a\circ f_a^{-1} : \mathbf{Z}_{a,+\infty} \to \mathbf{Z}_{-\infty,a}$;

where, for every $x\in\mathbf{Z}_{a,+\infty}$,

$$h_{a,a}(x)=g_a(f_a^{-1}(x))=\begin{cases} a-2\cdot f_a^{-1}(x) & , \text{if } f_a^{-1}(x)\geq 0 \\ a+2\cdot f_a^{-1}(x)+1, & \text{if } f_a^{-1}(x)<0 \end{cases}=2\cdot a-x$$

$$=a-(x-a),$$

and

$h_a^{-1}=f_a\circ g_a^{-1} : \mathbf{Z}_{-\infty,a} \to \mathbf{Z}_{a,+\infty}$;

where, for every $x\in\mathbf{Z}_{-\infty,a}$,

$$h_a^{-1}(x)=f_a(g_a^{-1}(x))=\begin{cases} a+2\cdot g_a^{-1}(x) & , \text{if } g^{-1}(x)\geq 0 \\ a-2\cdot g_a^{-1}(x)-1, & \text{if } g^{-1}(x)<0 \end{cases}=2\cdot a-x$$

$$=a+(a-x).$$

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6. Findings

Therefore, we answered the two questions in Paragraph 3. Thus, for any number a, b$\in$**Z** there are two pairs of laws of internal composition on the sets $\mathbf{Z}_{-\infty,a}$ and $\mathbf{Z}_{b,+\infty}$, let's say „♣” and „♦”, respectively „♥” and „♠”, so that ($\mathbf{Z}_{-\infty,a}$,♣,♦) and ($\mathbf{Z}_{b,+\infty}$,♥,♠) become commutative rings isomorphic to the ring (**Z**,+,·).

Concretely, on the set of integers not more than 3, $\mathbf{Z}_{-\infty,3}$, and on the set of integers at least equal to 5, $\mathbf{Z}_{5,+\infty}$, we can define two pairs of laws of internal composition so that let's say „♣” and „♦”, respectively „♥” and „♠”, so that ($\mathbf{Z}_{-\infty,3}$,♣,♦) and ($\mathbf{Z}_{5,+\infty}$,♥,♠) become commutative rings isomorphic to the ring (**Z**,+,·).

7. Conclusion

Everything we have done above can be rethought and put differently.

Let be a, b$\in$**Z**. We note with:

$\mathbf{Z}_{-\infty,a}=\{x\in\mathbf{Z} \mid x\le a\}$ and $\mathbf{Z}_{b,+\infty}=\{x\in\mathbf{Z} \mid x\ge b\}$.

Then the functions:

$g_a : \mathbf{Z} \to \mathbf{Z}_{-\infty,a}$ and $f_b : \mathbf{Z} \to \mathbf{Z}_{b,+\infty}$,

defined by:

$$f_b(x)=\begin{cases} b-2\cdot x & \text{, if } x\le 0 \\ b+2\cdot x-1, & \text{if } x>0 \end{cases} \quad \text{and} \quad g_a(x)=\begin{cases} a+2\cdot x & \text{, if } x\le 0 \\ a-2\cdot x+1, & \text{if } x>0 \end{cases}$$

are bijections, and their inverses are functions:

$g_a^{-1} : \mathbf{Z}_{-\infty,a} \to \mathbf{Z}$ and $f_b^{-1} : \mathbf{Z}_{b,+\infty} \to \mathbf{Z}$,

defined by:

$$g_a^{-1}(x)=\begin{cases} -\dfrac{a-x}{2} & \text{, if } x\text{ - a is even} \\ \dfrac{(a\text{ - }x)+1}{2}, & \text{if } x\text{ - a is odd} \end{cases} \quad \text{and} \quad f_b^{-1}(x)=\begin{cases} \dfrac{b-x}{2} & \text{, if } x\text{ - b is even} \\ \text{-}\dfrac{(b\text{ - }x)-1}{2}, & \text{if } x\text{ - b is odd} \end{cases},$$

which, again according to Vălcan (2019), shows that:

$\mathbf{Z} \sim \mathbf{Z}_{b,+\infty}$ and $\mathbf{Z} \sim \mathbf{Z}_{-\infty,a}$,

whence it follows that:

$\mathbf{Z}_{b,+\infty} \sim \mathbf{Z}_{-\infty,a}$;

the bijection that accomplishes this is:

$h_{a,b}=f_b\circ g_a^{-1} : \mathbf{Z}_{-\infty,a} \to \mathbf{Z}_{b,+\infty}$;

where, for every $x\in\mathbf{Z}_{-\infty,a}$,

$$f_b(x)=\begin{cases} b-2\cdot x & \text{, if } x\le 0 \\ b+2\cdot x-1, & \text{if } x>0 \end{cases} \quad \text{and} \quad g_a(x)=\begin{cases} a+2\cdot x & \text{, if } x\le 0 \\ a-2\cdot x+1, & \text{if } x>0 \end{cases}$$

$$h_{a,b}(x)=f_b(g_a^{-1}(x))=\begin{cases} b-2\cdot g_a^{-1}(x) & \text{, if } g_a^{-1}(x)\le 0 \\ b+2\cdot g_a^{-1}(x)-1, & \text{if } g_a^{-1}(x)>0 \end{cases}=a+b-x$$

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$=b+(a-x)$,

and

$h_{a,b}^{-1}=g_a \circ f_b^{-1}: \mathbf{Z}_{b,+\infty} \to \mathbf{Z}_{-\infty,a}$;

where, for every $x \in \mathbf{Z}_{b,+\infty}$,

$$h_{a,b}^{-1}(x)=g_a(f_b^{-1}(x))=\begin{cases} a+2\cdot f_a^{-1}(x) & \text{, if } f_a^{-1}(x)\le 0 \\ a-2\cdot f_a^{-1}(x)+1, & \text{if } f_a^{-1}(x)>0 \end{cases}=a+b-x$$

$=a-(x-b)$.

Therefore the following diagram (C) is commutative (see figure 3):

$$\begin{tikzcd} \mathbf{Z} \arrow[r, "f_b"] \arrow[d, "g_a"'] & \mathbf{Z}_{b,+\infty} \\ \mathbf{Z}_{-\infty,a} \arrow[ur, "h_{a,b}=f_b\circ g_a^{-1}"'] & \end{tikzcd} \qquad \text{(C)}$$

Diagram (C)

Moreover, the results in Paragraph 5 can be obtained in exactly the same way, also starting from these bijections, so that the diagram (C), above, is a commutative diagram of commutative rings. We leave this to the reader who is attentive and interested in these issues.

Now we can say that the next diagram of rings is commutative, that is all the rings in this diagram are isomorphic. Moreover, all commutative rings of integers, determined in (Vălcan, 2020) are isomorphic with those determined in this paper.

So we have the following commutative diagram (D):

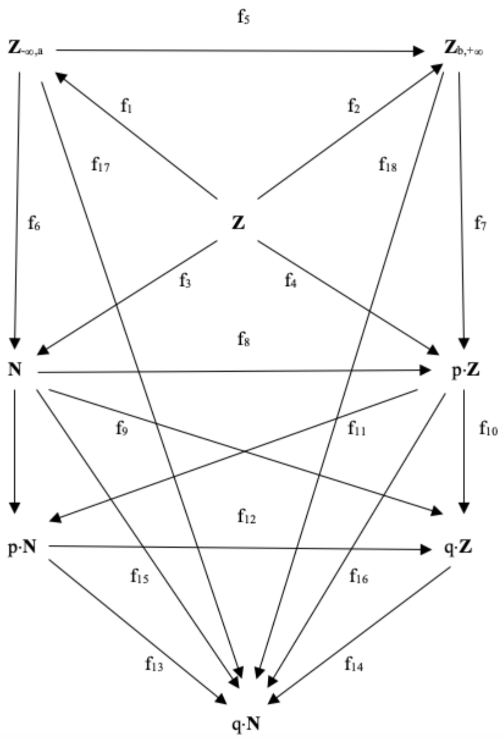

Diagram (D)

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The isomorphisms in the diagram above are as follows (see figure 4):

$f_1 : \mathbf{Z} \to \mathbf{Z}_{-\infty,a}$, $f_1(x)=\begin{cases} a-2\cdot x & \text{, if } x \ge 0 \\ a+2\cdot x+1 & \text{, if } x<0 \end{cases}$,

$f_1^{-1} : \mathbf{Z}_{-\infty,a} \to \mathbf{Z}$, $f_1^{-1}(x)=\begin{cases} \dfrac{a-x}{2} & \text{, if } x-a \text{ is even} \\ -\dfrac{(a-x)+1}{2} & \text{, if } x-a \text{ is odd} \end{cases}$,

$f_2 : \mathbf{Z} \to \mathbf{Z}_{b,+\infty}$, $f_2(x)=\begin{cases} b+2\cdot x & \text{, if } x \ge 0 \\ b-2\cdot x-1 & \text{, if } x<0 \end{cases}$,

$f_2^{-1} : \mathbf{Z}_{b,+\infty} \to \mathbf{Z}$, $f_2^{-1}(x)=\begin{cases} \dfrac{x-b}{2} & \text{, if } x-b \text{ is even} \\ -\dfrac{(x-b)+1}{2} & \text{, if } x-b \text{ is odd} \end{cases}$

$f_3 : \mathbf{Z} \to \mathbf{N}$, $f_3(x)=\begin{cases} 2\cdot x & \text{, if } x \ge 0 \\ -2\cdot x-1 & \text{, if } x<0 \end{cases}$,

$f_3^{-1} : \mathbf{N} \to \mathbf{Z}$, $f_3^{-1}(x)=\begin{cases} \dfrac{x}{2} & \text{, if } x \text{ is even} \\ -\dfrac{x+1}{2} & \text{, if } x \text{ is odd} \end{cases}$,

$f_4 : \mathbf{Z} \to p\cdot\mathbf{Z}$, $f_4(x)=p\cdot x$,

$f_4^{-1} : p\cdot\mathbf{Z} \to \mathbf{Z}$, $f_4^{-1}(p\cdot x)=x$,

$f_5 : \mathbf{Z}_{-\infty,a} \to \mathbf{Z}_{b,+\infty}$, $f_5(x)=b+(a-x)$,

$f_5^{-1} : \mathbf{Z}_{b,+\infty} \to \mathbf{Z}_{-\infty,a}$, $f_5^{-1}(x)=a-(x-b)$,

$f_6=f_3\circ f_1^{-1} : \mathbf{Z}_{-\infty,a} \to \mathbf{N}$, $f_6(x)=a-x$,

$f_6^{-1} : \mathbf{N} \to \mathbf{Z}_{-\infty,a}$, $f_6^{-1}(x)=a-x$,

$f_7=f_4\circ f_2^{-1} : \mathbf{Z}_{b+\infty} \to p\cdot\mathbf{Z}$, $f_7(x)=\begin{cases} p\cdot\dfrac{x-b}{2} & \text{, if } x-b \text{ is even} \\ -p\cdot\dfrac{(x-b)+1}{2} & \text{, if } x-b \text{ is odd} \end{cases}$,

$f_7^{-1} : p\cdot\mathbf{Z} \to \mathbf{Z}_{b+\infty}$, $f_7^{-1}(p\cdot x)=\begin{cases} b+2\cdot x & \text{, if } p\cdot x \ge 0 \\ b-2\cdot x-1 & \text{, if } p\cdot x<0 \end{cases}$,

$f_8 : \mathbf{N} \to p\cdot\mathbf{Z}$, $f_8(x)=\begin{cases} p\cdot\dfrac{x}{2} & \text{, if } x \text{ is even} \\ -p\cdot\dfrac{(x+1)}{2} & \text{, if } x \text{ is odd} \end{cases}$,

$f_8^{-1} : p\cdot\mathbf{Z} \to \mathbf{N}$, $f_8^{-1}(p\cdot x)=\begin{cases} 2\cdot x & \text{, if } p\cdot x \ge 0 \\ -2\cdot x-1 & \text{, if } p\cdot x<0 \end{cases}$,

$f_9 : \mathbf{N} \to p\cdot\mathbf{N}$, $f_9(x)=p\cdot x$,

$f_9^{-1} : p\cdot\mathbf{N} \to \mathbf{N}$, $f_9^{-1}(p\cdot x)=x$,

$f_{10} : p\cdot\mathbf{Z} \to q\cdot\mathbf{Z}$, $f_{10}(p\cdot x)=q\cdot x$,

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$f_{10}^{-1} : q\cdot\mathbf{Z} \to p\cdot\mathbf{Z}$, $f_{10}^{-1}(q\cdot x)=p\cdot x$,

$f_{11}=f_{10}\circ f_8 : \mathbf{N} \to q\cdot\mathbf{Z}$, $f_{11}(x)=\begin{cases} q\cdot\frac{x}{2}, & \text{if } x \text{ is even} \\ -q\cdot\frac{(x+1)}{2}, & \text{if } x \text{ is odd}\end{cases}$,

$f_{11}^{-1} : q\cdot\mathbf{Z} \to \mathbf{N}$, $f_{11}^{-1}(q\cdot x)=\begin{cases} 2\cdot x, & \text{if } q\cdot x \ge 0 \\ -2\cdot x-1, & \text{if } q\cdot x < 0\end{cases}$,

$f_{12}=f_{11}\circ f_9^{-1} : p\cdot\mathbf{N} \to q\cdot\mathbf{Z}$, $f_{12}(p\cdot x)=\begin{cases} q\cdot\frac{x}{2}, & \text{if } x \text{ is even} \\ -q\cdot\frac{(x+1)}{2}, & \text{if } x \text{ is odd}\end{cases}$,

$f_{12}^{-1} : q\cdot\mathbf{Z} \to p\cdot\mathbf{N}$, $f_{12}^{-1}(q\cdot x)=\begin{cases} 2\cdot p\cdot x, & \text{if } q\cdot x \ge 0 \\ -p\cdot(2\cdot x+1), & \text{if } q\cdot x < 0\end{cases}$,

$f_{13} : p\cdot\mathbf{N} \to q\cdot\mathbf{N}$, $f_{13}(p\cdot x)=q\cdot x$,

$f_{13}^{-1} : q\cdot\mathbf{N} \to p\cdot\mathbf{N}$, $f_{13}^{-1}(q\cdot x)=p\cdot x$,

$f_{14}= f_{13}\circ f_{12}^{-1} : q\cdot\mathbf{Z} \to q\cdot\mathbf{N}$, $f_{14}(q\cdot x)=\begin{cases} 2\cdot q\cdot x, & \text{if } q\cdot x \ge 0 \\ -q\cdot(2\cdot x+1), & \text{if } q\cdot x < 0\end{cases}$,

$f_{14}^{-1} : q\cdot\mathbf{N} \to q\cdot\mathbf{Z}$, $f_{14}^{-1}(q\cdot x)=\begin{cases} q\cdot\frac{x}{2}, & \text{if } x \text{ is even} \\ -q\cdot\frac{(x+1)}{2}, & \text{if } x \text{ is odd}\end{cases}$,

$f_{15} : \mathbf{N} \to q\cdot\mathbf{N}$, $f_{15}(x)=q\cdot x$,

$f_{15}^{-1} : q\cdot\mathbf{N} \to \mathbf{N}$, $f_{15}^{-1}(q\cdot x)=x$,

$f_{16}=f_{14}\circ f_{10} : p\cdot\mathbf{Z} \to q\cdot\mathbf{N}$, $f_{16}(p\cdot x)=\begin{cases} 2\cdot q\cdot x, & \text{if } p\cdot x \ge 0 \\ -q\cdot(2\cdot x+1), & \text{if } p\cdot x < 0\end{cases}$,

$f_{16}^{-1} : q\cdot\mathbf{N} \to p\cdot\mathbf{Z}$, $f_{16}^{-1}(q\cdot x)=\begin{cases} p\cdot\frac{x}{2}, & \text{if } x \text{ is even} \\ -p\cdot\frac{(x+1)}{2}, & \text{if } x \text{ is odd}\end{cases}$,

$f_{17}=f_{15}\circ f_6 : \mathbf{Z}_{-\infty,a} \to q\cdot\mathbf{N}$, $f_{17}(x)=q\cdot(a-x)$,

$f_{17}^{-1} : q\cdot\mathbf{N} \to \mathbf{Z}_{-\infty,a}$, $f_{17}^{-1}(q\cdot x)=a-x$,

$f_{18}=f_{16}\circ f_7 : \mathbf{Z}_{b+\infty} \to q\cdot\mathbf{N}$, $f_{18}(x)=q\cdot(x-b)$,

$f_{18}^{-1} : q\cdot\mathbf{N} \to \mathbf{Z}_{b,+\infty}$, $f_{18}^{-1}(q\cdot x)=b+x$.

At the end of this paper, we specify that the bijection from **Z** to **N** can be considered as a function:

$f : \mathbf{Z} \to \mathbf{N}$, $f(x)=\begin{cases} -2\cdot x, & \text{if } x \le 0 \\ 2\cdot x-1, & \text{if } x > 0\end{cases}$,

in which case:

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$f^{-1} : \mathbf{N} \to \mathbf{Z}$, $f^{-1}(x)=\begin{cases} -\dfrac{x}{2}, & \text{if } x \text{ is even} \\ \dfrac{x+1}{2}, & \text{if } x \text{ is odd} \end{cases}$.

The determination of the two operations on N is also left to the attentive and interested reader of these issues, to which we yield that we have formed a good image about the determination of the commutative rings of integers, isomorphic between them and isomorphic with the commutative ring $(\mathbf{Z},+,\cdot)$.

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