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A FIELD OF FUNCTIONS ISOMORPHIC TO THE FIELD OF REAL NUMBERS

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Abstract

Mathematics is a science that cannot be defined. However, we can say that Mathematics is dealing with solving problems of a certain type, but it is not limited to this. Knowing Mathematics does not mean knowing how to solve problems, and vice versa, knowing how to solve problems is not to know Mathematics. Mathematics is a way of thinking, it is an art, a global cultural movement. She surprises everything with her reasoning and her problems. That's what he does here. Here's how: the problem of determining isomorphic structures is difficult for pupils, students or teachers, although it can be approached from three perspectives: inductive, deductive and analogous. On the other hand, hyperbolic functions and their properties are also poorly known. In this paper, using the hyperbolic functions and their properties, we will determine a set of functions of real variable, which we then endow it with two operations, to give it a structure o field isomorphic to the field $(\mathbf{R},+,\cdot)$ and implicitly with all the fields isomorphic to this field. It will result, then, that any real number can be represented as a function. Moreover, because there are the fields of matrix which are isomorphic to the field of real numbers, will result that there are matrices that can be represented as a function determined by us in this paper.

Keywords: Mathematics, Didactics of Mathematics, commutative field, isomorphism.

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1. Introduction

It is known that today Mathematics has penetrated all fields of human activity. The role of Mathematics as a subject of school education is very important. Mathematics contributes to the intellectual formation of man and constitutes a basic element in human culture.

Mathematics turns out to be, first, a totalizing interpretation of the Universe, a conception of the world as a whole. The picture of reality that science provides us is made up of fragments of knowledge, refers to "*portions*" of existence (which explains the power of intervention on the realities of science). However generous the generations of science are, they give us knowledge a parcel of reality, we propose effective solutions to partial problems, not taking into account, by their very nature, existence as totality. That's why, it becomes nonsense any attempt to raise science, as a result of heterogeneous knowledge at the rank of a worldview, able to propose far-reaching ideals to those which Mathematics proposes.

The image that Mathematics proposes to the world it is not the writing of the data, but by their interpretation, that is, by its own theoretical effort, which leads to the formulation of statements which have as object the Universe, the infinite, etc. Underlining the specific mode of being of Mathematics by reference to science and, in particular, of its integrative valences, engages in the lesson of Mathematics in an extremely generous horizon, which the teacher cannot neglect. It's about another degree of generality, who employs Mathematics in a perpetual aspiration to absolute and which is required to be emphasized with consistency. Also, adherence to a mathematical or other theory not only problems of pure thinking and logical consistency, but also of conviction, of attitude, of adhesion to certain ideals and values. And all this way of being of Mathematics it is required to be imposed in the steps taken by the teacher in teaching, to configure the possibility of another horizon in students' consciousness, broader and more generous than their experience from the real recording through the data provided by the study of the various disciplines (Vălcan, 2013).

The totalizing vision, specifies Mathematics, in the attempt to explain the Universe as a whole and to coordinate the multitude of values which governs man's life, is constituted not only through totalization scientific knowledge, but also through totalization specifies mathematical knowledge, constituted concomitant on the data knowing discoverer, experience lived and on collective human action, an image of the world as a whole. Thus, Mathematics is distinctive (compared to other science) and by another reference, tending, even when addressing the same aspects of reality as well as science to report on the issue to the ontological status of man. It's important for any adolescent, we believe, to understand that from harnessing knowledge in their human meaning it result the possibility of a vision of human destiny, the meaning of its existence, vision what drives a moral attitude, at least.

This explains one more time, why, as well as scientific language, the mathematical one is susceptible to univocal interpretations. We show (and) in this paper that he assumes a justification in relation to a value scale:

Mathematics is such a reflexive floor to which a broad explanatory synthesis is conjugated with a human reaction to the world (Rus & Munteanu, 1998).

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2. Problem Statement

We can say, without fear that we are wrong, that all the content of Mathematics (both scientific and didactic) is organized on algebraic structures, and some such structures are, in addition, and vector or topological spaces. According to Purdea and Pic (1977), the starting point for such structural approaches is, usually, one of the known, of numbers, functions, vectors, matrices; the base being the field of the complex numbers ($\mathbf{C}$,+,·), with its substructures of natural numbers, integers, rational or real.

Here's what you should know in addition in such cases (Vălcan, 2017):

Corollary 2.6: *If (A,+,·) is a (commutative) ring / field and B is a set equipotent with the set A, then there are two laws of internal composition on B, which we note with „○" and „•", such that (R,○,•) is a (commutative) ring / field.*

From here, we deduce that:

Corollary 2.7: *If (A,+,·) is a (commutative) ring / field, and H and K are two sets equipotent with A, then, on each from the sets H and K, there are two laws of internal composition which determines on H, respectively on K, a structure of (commutative) ring / field, each isomorph to the ring / the field A, such that the following diagram (of isomorphic rings / fields) is commutative*:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & H \arrow[dl] \\ K. & \end{tikzcd} \tag{2.11}$$

The corollary above shows us, very clear, that everything that is happening / is valid in the ring / field A happens / is also valid in each of the fields H, respectively K.

So, all properties and calculation rules from the field ($\mathbf{R}$,+,·) are valid in all isomorphic fields to it; so also in: $\mathbf{R}_{-\infty,a}$, $\mathbf{R}_{b,c}$ and $\mathbf{R}_{d,+\infty}$, defined in Vălcan (2017), with a, b, c and d$\in\mathbf{R}$.

3. Research Questions

In our research we will try to find answers to the following questions:

-There are structures of fields defined on sets of functions and which are isomorphic to the field of real numbers, ($\mathbf{R}$,+,·)?

-How can these structures be identified?

-Can hyperbolic functions to generate such structures?

4. Purpose of the Study

Further, we will try to answer the three questions above.

So, we will show that for any real number a, the values of the hyperbolic functions sh and ch in point a (which we will simply note with sha and cha - see Vălcan (2016) generates a set of functions:

$F=\{f_a : \mathbf{R} \to \mathbf{R} \mid a\in\mathbf{R}\}$,

which, then, can be defined two laws of internal composition, which we denote with „∘", and „∗", such that (F,∘,∗) becomes a field isomorphic to the field of real numbers, ($\mathbf{R}$,+,·).

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5. Research Methods

In this paragraph, we will prove, in two ways, what I said in the preceding paragraph.

To understanding the result below, suppose the reader is familiar with the notions that work here: function, internal composition law, groupoid / stable / closed part, semigroup, monoid, group, ring or field, isomorphism, sine and hyperbolic cosine, as defined / specified in mathword.

The main result of this work is the following:

Theorem: *There is a field of functions that is isomorphic to the field of real numbers, $(\mathbf{R},+\cdot)$.*

***Proof*:** Let be $a\in\mathbf{R}$ and the function:

$$f_a : \mathbf{R} \to \mathbf{R},$$

defined by: for every $x\in\mathbf{R}$,

(1) $f_a(x)=x\cdot cha+\sqrt{1+x^2}\cdot sha.$

If:

$$F=\{f_a \mid a\in\mathbf{R}\},$$

then $(F,\circ,*)$ is a field isomorphic to the field $(\mathbf{R},+,\cdot)$; where „$\circ$" is the usual composition of functions, and „$*$" is the following operation defined on F thus: if a, $b\in\mathbf{R}$ and f_a, $f_b\in F$, then:

(2) $f_a*f_b=f_{a\cdot b}\in F,$

where,

$$f_{a\cdot b} : \mathbf{R}\to\mathbf{R},$$

defined by: for every $x\in\mathbf{R}$,

(3) $f_{a\cdot b}(x)=x\cdot ch(a\cdot b)+\sqrt{1+x^2}\cdot sh(a\cdot b).$

Indeed, if a, $b\in\mathbf{R}$ and f_a, $f_b\in F$, then, according to the equalities (2.2) and (2.4) from Vălcan (2016), for every $x\in\mathbf{R}$,

(4) $(f_a\circ f_b)(x)=(f_a(f_b(x))=f_b(x)\cdot cha+\sqrt{1+(f_b(x))^2}\cdot sha$

$$=(x\cdot chb+\sqrt{1+x^2}\cdot shb)\cdot cha+\sqrt{1+(x\cdot chb+\sqrt{1+x^2}\cdot shb)^2}\cdot sha$$

$$=x\cdot chb\cdot cha+\sqrt{1+x^2}\cdot shb\cdot cha$$

$$+\sqrt{1+x^2\cdot ch^2b+2\cdot x\cdot\sqrt{1+x^2}\cdot shb\cdot chb+(1+x^2)\cdot sh^2b}\cdot sha$$

$$=x\cdot chb\cdot cha+\sqrt{1+x^2}\cdot shb\cdot cha$$

$$+\sqrt{1+x^2\cdot ch^2b+2\cdot x\cdot\sqrt{1+x^2}\cdot shb\cdot chb+sh^2b+x^2\cdot sh^2b}\cdot sha$$

$$=x\cdot chb\cdot cha+\sqrt{1+x^2}\cdot shb\cdot cha$$

$$+\sqrt{(1+sh^2b)+x^2\cdot(ch^2b+sh^2b)+2\cdot x\cdot\sqrt{1+x^2}\cdot shb\cdot chb}\cdot sha$$

$$=x\cdot chb\cdot cha+\sqrt{1+x^2}\cdot shb\cdot cha$$

$$+\sqrt{ch^2b+x^2\cdot ch^2b+2\cdot x\cdot\sqrt{1+x^2}\cdot shb\cdot chb+x^2\cdot sh^2b}\cdot sha$$

$$=x\cdot chb\cdot cha+\sqrt{1+x^2}\cdot shb\cdot cha$$

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$$+\sqrt{(1+x^2)\cdot ch^2b+2\cdot x\cdot\sqrt{1+x^2}\cdot shb\cdot chb+x^2\cdot sh^2b}\cdot sha$$

$$=x\cdot chb\cdot cha+\sqrt{1+x^2}\cdot shb\cdot cha+\sqrt{\left(\sqrt{1+x^2}\cdot chb+x\cdot shb\right)^2}\cdot sha$$

$$=x\cdot chb\cdot cha+\sqrt{1+x^2}\cdot shb\cdot cha+(x\cdot shb+\sqrt{1+x^2}\cdot chb)\cdot sha$$

$$=x\cdot chb\cdot cha+\sqrt{1+x^2}\cdot shb\cdot cha+x\cdot shb\cdot sha+\sqrt{1+x^2}\cdot chb\cdot sha$$

$$=x\cdot(chb\cdot cha+shb\cdot sha)+\sqrt{1+x^2}\cdot(chb\cdot sha+shb\cdot cha)$$

$$=x\cdot ch(a+b)+\sqrt{1+x^2}\cdot sh(a+b)=x\cdot ch(a+b)+\sqrt{1+x^2}\cdot sh(a+b)$$

$$=f_{a+b}(x).$$

Therefore, the equality (2) holds:

$$f_a\circ f_b=f_{a+b}\in F,$$

which shows that the set F is closed in relation to the function composing operation, that is, this operation is an internal composition law on the set F, or, otherwise, the set F is a stable part (closed) of it:

$$\mathbf{R}^{\mathbf{R}}=\{f:\mathbf{R}\to\mathbf{R}\}$$

in relation to this operation. Above we used that, for every $x\in\mathbf{R}$ and every $b\in\mathbf{R}$,

(5) $x\cdot shb+\sqrt{1+x^2}\cdot chb\geq 0.$

Indeed, let be $b\in\mathbf{R}$ and consider the function:

$$g_b:\mathbf{R}\to\mathbf{R},$$

defined by: for every $x\in\mathbf{R}$,

$$g_b(x)=x\cdot shb+\sqrt{1+x^2}\cdot chb.$$

Then, for every $x\in\mathbf{R}$:

$$(g_b(x))'=shb+\frac{x}{\sqrt{1+x^2}}\cdot chb,$$

$$(g_b(x))''=\frac{\sqrt{1+x^2}-x\cdot\frac{x}{\sqrt{1+x^2}}}{1+x^2}\cdot chb=\frac{1+x^2-x^2}{(1+x^2)\cdot\sqrt{1+x^2}}\cdot chb=\frac{1}{(1+x^2)\cdot\sqrt{1+x^2}}\cdot chb,$$

and

$$\lim_{x\to-\infty}(g_b(x))'=-chb, \qquad \lim_{x\to+\infty}(g_b(x))'=chb,$$

$$\lim_{x\to-\infty}g_b(x)=\lim_{x\to-\infty}(x\cdot shb+\sqrt{1+x^2}\cdot chb)=\lim_{x\to+\infty}(\sqrt{1+x^2}\cdot chb-x\cdot shb)$$

$$=\lim_{x\to+\infty}x\cdot\left(\frac{\sqrt{1+x^2}}{x}\cdot chb-shb\right)=+\infty,$$

because, for every $x>0$ and every $b\in\mathbf{R}$,

$$\frac{\sqrt{1+x^2}}{x}>1 \qquad \text{and} \qquad chb>shb,$$

that is:

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$$\frac{\sqrt{1+x^2}}{x}\cdot chb\text{-}shb>0.$$

$$\lim_{x\to+\infty} g_b(x)=\lim_{x\to+\infty}(x\cdot shb+\sqrt{1+x^2}\cdot chb)=\lim_{x\to+\infty} x\cdot\left(\frac{\sqrt{1+x^2}}{x}\cdot chb+shb\right)=+\infty,$$

because, for every x>0 and every $b\in\mathbf{R}$,

$$\frac{\sqrt{1+x^2}}{x}>1 \qquad \text{and} \qquad chb>-shb,$$

that is:

$$\frac{\sqrt{1+x^2}}{x}\cdot chb+shb>0.$$

Now we can present the following table of variation of the function g_b:

x	$-\infty$	0	$+\infty$
$(g_b(x))''$	+ + + + + + + + + + + + + + + + + +	+ + + + + + + + + + + + + + + + + +	
$(g_b(x))'$	-chb $\nearrow$	0 $\searrow$	+chb
$(g_b(x))'$	- -	0 + + + + + + + + + + + + + + + + + +	
$g_b(x)$	$+\infty$ $\searrow$	1 $\nearrow$	$+\infty$

From this table it follows that, for every $x\in\mathbf{R}$ and every $b\in\mathbf{R}$,

$$g_b(x)=x\cdot shb+\sqrt{1+x^2}\cdot chb\geq 1.$$

Equality (5) can also prove this way: we notice that it is equivalent to: for every $x\in\mathbf{R}$ and for every $b\in\mathbf{R}$,

$$\sqrt{1+x^2}\cdot\frac{e^b+e^{-b}}{2}+x\cdot\frac{e^b-e^{-b}}{2}\geq 0,$$

that is: for every $x\in\mathbf{R}$ and for every $b\in\mathbf{R}$,

$$e^b\cdot(\sqrt{1+x^2}+x)+e^{-b}\cdot(\sqrt{1+x^2}-x)\geq 0,$$

what is true, because each parenthesis and its coefficient is positive. We are now returning to the original problem. Composition of functions is (always) an associative operation, but, according to the above, we can easily check this, for our case: if a, b, $c\in\mathbf{R}$ and $f_a, f_b, f_c\in F$, then,

(6) $(f_a\circ f_b)\circ f_c=f_{a+b}\circ f_c=f_{(a+b)+c}=f_{a+(b+c)}$
$=f_a\circ(f_b\circ f_c)$.

On the other hand, if a, $b\in\mathbf{R}$ and $f_a, f_b\in F$, then,

(7) $f_a\circ f_b=f_{a+b}=f_{b+a}$
$=f_b\circ f_a$,

which shows that the compositional law „$\circ$" is commutative on F, although, in general, this does not happen on the $\mathbf{R}^{\mathbf{R}}$. Before verifying the existence of the neutral (identity) element and the symmetric (inverse) of each element $f_a\in F$, observe that, if a, $b\in\mathbf{R}$ and $f_a, f_b\in F$, then,

(8) $f_a=f_b \Leftrightarrow a=b$.

Indeed, for every $x\in\mathbf{R}$, the following equivalences hold:

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$f_a=f_b \Leftrightarrow f_a(x)=f_b(x) \Leftrightarrow x\cdot cha+\sqrt{1+x^2}\cdot sha=x\cdot chb+\sqrt{1+x^2}\cdot shb.$

For:

$x=0$

last equality leads to:

$sha=shb,$

which, according to Property 23) from Vălcan (2016), is equivalent to:

$a=b.$

Now we verify the existence of the neutral (identity) element in the semigroup $(F,\circ)$. We assume that it exists; let's note it with $f_e\in F$, with $e\in\mathbf{R}$. Then, for every $a\in\mathbf{R}$ and $f_a\in F$,

(9) $f_a\circ f_e=f_e\circ f_a=f_{a+e}$

$=f_a.$

From equivalence (8) and equalities (9), it follows that:

$a+e=a,$ that is: $e=0$

and $f_0\in F$ is the neutral (identity) element in relation to the law „$\circ$"; the function f_0 is defined by: for every $x\in\mathbf{R}$,

(10) $f_0(x)=x\cdot ch0+\sqrt{1+x^2}\cdot sh0=x$

$=1_{\mathbf{R}}(x).$

When determining the neutral element on F we can proceed in another way: we know that the identical function $1_{\mathbf{R}}$ is (in general!) a neutral (identity) element to the composition of the functions and we notice that the above equalities hold, which shows that $1_{\mathbf{R}}\in F$. Now we show that, for every $a\in\mathbf{R}$, the element $f_a\in F$ has a symmetric (inverse) in F, in relation to the law „$\circ$". Indeed, if $a'\in\mathbf{R}$ and $f_{a'}\in F$ such that:

(11) $f_0=f_a\circ f_{a'}=f_{a+a'}=f_{a'+a}$

$=f_{a'}\circ f_a,$

then, from the equivalence (8), obtain that:

$a'=-a$

and $f_{-a}\in F$ is the symmetric (inverse) of element $f_a\in F$; the function f_{-a} is defined by: for every $x\in\mathbf{R}$,

(12) $f_{-a}(x)=x\cdot ch(-a)+\sqrt{1+x^2}\cdot sh(-a)=x\cdot cha-\sqrt{1+x^2}\cdot sha;$

according to the equalities (1.9) and (1.10) from Vălcan (2016). Therefore, the composition of functions determines on the set F a commutative group structure. Now we will show that $(F,*)$ is a commutative monoid. Indeed, from the definition of „$*$" it is noted that this is an internal composition law on F, that is the set F is closed in relation to the operation „$*$",or, in other words, F is a stable part of $\mathbf{R}^{\mathbf{R}}$ in relation to the law „$*$".The associativity of operation „$*$" results immediately, because: if a, b, $c\in\mathbf{R}$ and f_a, f_b, $f_c\in F$, then,

(13) $(f_a*f_b)*f_c=f_{a\cdot b}*f_c=f_{(a\cdot b)\cdot c}=f_{a\cdot(b\cdot c)}$

$=f_a*(f_b*f_c).$

On the other hand, if a, $b\in\mathbf{R}$ and f_a, $f_b\in F$, then,

(14) $f_a*f_b=f_{a\cdot b}=f_{b\cdot a}$

$=f_b*f_a,$

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which shows that the compositional law „∗"is commutative on F. Now we verify the existence of the neutral (identity) element in the semigroup (F,∗). We assume that it exists; let's note it with $f_{e'}\in F$, where $e'\in\mathbf{R}$. Then, for every $a\in\mathbf{R}$ and $f_a\in F$,

(15) $f_a*f_{e'}=f_{e'}*f_a=f_{a\cdot e'}$

$=f_a$.

From the equivalences (15) and (8) it follows that:

$a\cdot e'=a$, that is: $e'=1$

and $f_1\in F$ the neutral (identity) element in relation to the law „∗"; the function f_1 is defined by: for every $x\in\mathbf{R}$,

(16) $f_1(x)=x\cdot ch1+\sqrt{1+x^2}\cdot sh1$.

Now we show that, for every $a\in\mathbf{R}^*$, the element $f_a\in F\setminus\{f_0\}$ has a symmetric (inverse) in F, in relation to the law „∗". Indeed, if $a''\in\mathbf{R}^*$ and $f_{a''}\in F$ such that:

(17) $f_1=f_a*f_{a''}=f_{a\cdot a''}=f_{a''\cdot a}=f_{a''}*f_a$,

then, from the equivalences (17) and (8), it follows that:

$a''=a^{-1}$

and $f_{a^{-1}}\in F$ is the symmetric (inverse) of the element $f_a\in F$; the function $f_{a^{-1}}$ is defined by: for every $x\in\mathbf{R}$,

(18) $f_{a^{-1}}(x)=x\cdot ch(a^{-1})+\sqrt{1+x^2}\cdot sh(a^{-1})=x\cdot ch\dfrac{1}{a}-\sqrt{1+x^2}\cdot sh\dfrac{1}{a}$.

Therefore, the law „∗" determines on the set $F\setminus\{f_0\}$ also a commutative group structure. Because the two operations defined on the set F are defined by the operations of addition and multiplication of real numbers, the distributivity of the operation „∗" to operation „∘" will result from the distributivity of the multiplication of real numbers relative to the addition of these numbers. Indeed, if a, b, $c\in\mathbf{R}$ and f_a, f_b, $f_c\in F$, then,

(19) $(f_a\circ f_b)*f_c=f_{a+b}*f_c=f_{(a+b)\cdot c}=f_{a\cdot c+b\cdot c}$

$=(f_a*f_c)\circ(f_b*f_c)$;

respectively:

(20) $f_a*(f_b\circ f_c)=f_a*f_{b+c}=f_{a\cdot(b+c)}=f_{a\cdot b+a\cdot c}$

$=(f_a*f_c)\circ(f_a*f_c)$;

which shows distributivity to the right, respectively to the left, of the law „∗" to the law „∘".It follows from the above that (F,∘,∗) is a (commutative) field. The isomorphism reminded / asked in the statement is the following:

$\varphi : (\mathbf{R},+,\cdot) \to (F,\circ,*)$,

where, for every $a\in\mathbf{R}$,

(21) $\varphi(a)=f_a$.

Indeed, the injectivity of the application φ results from equivalence (8), and its surjectivity results from its definition and the set F. Thus, the application φ is bijective. On the other hand, if a, $b\in\mathbf{R}$ and f_a, $f_b\in F$, then,

(22) $\varphi(a+b)=f_{a+b}=f_a\circ f_b$

$=\varphi(a)\circ\varphi(b)$,

respectively:

(23) $\varphi(a\cdot b)=f_{a\cdot b}=f_a*f_b$

$=\varphi(a)*\varphi(b)$,

which shows that the application φ, as defined above, is morphism of fields; being also bijection, this is an isomorphism between the two fields.

***Otherwise*:** According those presented in Vălcan (2016), the function:

$sh : \mathbf{R} \to \mathbf{R}$,

defined by, for every $x\in\mathbf{R}$,

$$shx=\frac{e^x-e^{-x}}{2},$$

is bijective and, thus, it is invertible. So, for every $x\in\mathbf{R}$, there is a unique one $y\in\mathbf{R}$, such that:

$x=x(y)=shy$.

Now, from the relation (1) we obtain that:

(24) $f_a(x(y))=x\cdot cha+\sqrt{1+x^2}\cdot sha=shy\cdot cha+\sqrt{1+sh^2y}\cdot sha$

$=shy\cdot cha+chy\cdot sha=sh(y+a)$.

Then, if a, $b\in\mathbf{R}$ and f_a, $f_b\in F$, from the equality (24) above and the equalities (3.1) and (3.2) in Vălcan, (2017), we obtain that, for every $x(=x(y))\in\mathbf{R}$:

(25) $(f_a\circ f_b)(x(y))=f_a(f_b(x(y)))=f_a(sh(y+b))=sh(y+b)\cdot cha+\sqrt{1+sh^2(y+b)}\cdot sha$

$=sh(y+b)\cdot cha+ch(y+b)\cdot sha=sh(y+(b+a))=f_{b+a}(x(y))=f_{a+b}(x(y))$

$=(f_b\circ f_a)(x(y))$

and

(26) $(f_a*f_b)(x(y))=sh(y+a\cdot b)=f_{a\cdot b}(x(y))=f_{b\cdot a}(x(y))$

$=(f_a*f_b)(x(y))$.

The equalities above shows that the set F is closed to the two operations defined above and that they are commutative. And in this case, the associativity of the two operations results from the associativity of the operations of addition, respectively of multiplication of the real numbers, because, for every a, b, $c\in\mathbf{R}$ and every $x(y)\in\mathbf{R}$,

(27) $((f_a\circ f_b)\circ f_c)(x(y))=sh(y+(c+(b+a)))=sh(y+((c+b)+a))$

$=(f_a\circ(f_b\circ f_c))(x(y))$,

and,

(28) $((f_a*f_b)*f_c)(x(y))=(sin(y+(c\cdot b)\cdot a)=(sin(y+c\cdot(b\cdot a))$

$=(f_a*(f_b*f_c))(x(y))$.

The equivalence:

(29) $f_a=f_b \Leftrightarrow a=b$

follows from the injectivity of the function sh - see Property 22) in Vălcan (2016), and the existence of neutral (identity) elements in relation to the two operations results from the following equivalences,

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(30) $sh(y+a+e)=sh(y+a) \Leftrightarrow e=0$,

respectively

$sh(y+a\cdot e')=sh(y+a) \Leftrightarrow e'=1$.

Now, the fact that, for every $a\in\mathbf{R}$, the element $f_a\in F$ has a symmetric (inverse) in F, in relation to the law „$\circ$” it follows from the equivalence:

$sh(y+a+a')=sh(y) \Leftrightarrow a'=-a$,

and the fact that for every $a\in\mathbf{R}^*$, the element $f_a\in F$ a symmetric (inverse) in F, in relation to the law „$*$” it follows from the equivalence:

$$sh(y+a\cdot a'')=sh(y+a) \Leftrightarrow a''=\frac{1}{a}.$$

Finally, and in this case, the distributivity of operation „$*$” to operation „$\circ$” will result from the distributivity of multiplication of real numbers compared to the addition of these numbers. Therefore, of the above, it follows that $(F,\circ,*)$ is a (commutative) field, and the isomorphism reminded / asked in the statement is the same as in the first proof.

6. Findings

Therefore, I answered all the questions asked in Paragraph 3. So, any real number a can be represented as a function $f_a\in F$. For example, the sets **N**, **Z** and **Q** of natural numbers, integers and rational (from **R**) are represented in F by the following sets of functions:

$\varphi(\mathbf{N})=\{f_n \mid n\in\mathbf{N}\}$, $\varphi(\mathbf{Z})=\{f_m \mid m\in\mathbf{Z}\}$ and $\varphi(\mathbf{Q})=\{f_q \mid q\in\mathbf{Q}\}$.

Concretely, the number $5\in\mathbf{N}$ is represented in F by the function:

$f_5:\mathbf{R}\to\mathbf{R}$,

defined by: for every $x\in\mathbf{R}$,

$$f_5(x)=x\cdot ch5+\sqrt{1+x^2}\cdot sh5=x\cdot\frac{e^5+e^{-5}}{2}+\sqrt{1+x^2}\cdot\frac{e^5-e^{-5}}{2}.$$

7. Conclusion

It is known that, for every $n\in\mathbf{N}^*$, the set of matrices of order n:

$G_n=\{x\cdot I_n \mid x\in\mathbf{R}\}\subset M_n(\mathbf{R})$,

together with the usual addition and multiplication of the matrices, form a (commutative) field isomorphic to the field of real numbers, $(\mathbf{R},+,\cdot)$ – the verification is immediate. In conclusion, concidering those proven in Vălcan (2017) and those mentioned above, we can say that any real number can be represented as an element of any set of numbers: $\mathbf{R}_{-\infty,a}$, $\mathbf{R}_{b,c}$ and $\mathbf{R}_{d,+\infty}$, with a, b, c, and $d\in\mathbf{R}$, but also as a function of F or as a matrix of G_n.

More than that, $x\cdot I_n\in G_n$, is not the only matrix representation of the real number x; it can easily prove that the set of matrices:

$H=\{A(x) \mid x\in\mathbf{R}\}\subset M_3(\mathbf{R})$,

where, for every $x\in\mathbf{R}$,

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A(x)=$\begin{pmatrix} 1 & x & a\cdot x + a\cdot x^2 \\ 0 & 1 & 2\cdot a\cdot x \\ 0 & 0 & 1 \end{pmatrix}$, where a$\in$**R*** is fixed,

can be equipped with two laws of internal composition, say „•" and „⊥",, so that (H,•,⊥) becomes a (commutative) field isomorphic to the field (**R**,+,·).

Finally, mention that we wrote this paper in the context of a personal culture of teaching - learning Mathematics, culture developed and developed through individual study and experimentation in class, by following the steps in Becheanu et al., (1983) and trying to form and develop the reader interested in these issues, methodological competences of their approach.

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