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ON THE SIGNIFICANCE OF ARGUMENTATION IN SEARCHING FOR MATHEMATICAL PROOF

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A proof is an essay of sorts. …
the steps in a proof must follow logically from previous steps or
be justified by some other agreed-upon set of facts.
In addition to being valid, these steps must also fit coherently together to form a cogent argument.
Steven Abbott, *Understanding Analysis*, 2015

Abstract

In this paper, we argue that argumentation theory can be used to explore certain aspects of the development of discovery proof-events in time. Since argumentation is inseparable from the process of searching for a mathematical proof, a modified model of proof-events calculus, based on certain versions of argumentation theories primarily advanced by Toulmin and Pollock, can be used to this effect. We claim that the exchange of arguments and counterarguments set forward to clarify eventual gaps or implicit assumptions occurring in the course of a proof-event can be represented by appealing to argumentation theories. In this paper, a comparative analysis was carried out between the theory of proof-events and the theory of argumentation. The combination of these two theories enables us to represent controversial points in the process of searching for proof. By expanding the calculus of proof-events with the theory of argumentation we can take into consideration such moments as incomplete or even false purported proofs, intuitive ideas, correct or incorrect steps of reasoning, commentaries, etc. and represent them formally at the appropriate stages that take place during the evolution of a sequence of proof-events.

Keywords: Mathematical proof, proof-events, argument, argumentation theories.

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1. Introduction

In the above quotation (Abbott 2015, p. p. 8-9), the proof is compared with an essay made up of convincing arguments. A similar comparison was made in (Stefaneas, Vandoulakis, 2014, p.p. 117-119), where it is stated that "the structure of proof in mathematics is a special type of narrative structure", formulated in a certain semiotic code and organized in a complex hierarchical order. Such a structure is examined by means of semiotic discursive analysis. However, this concerns the outcome of the search for a mathematical proof, i.e. the exposition of a mathematical proof.

On the other hand, the value of argumentation in mathematics is not limited only to the formulation of a mathematical proof in the form of essay, but penetrates all the activity of searching for mathematical proof. In order to have a clearer understanding of the role of argumentation in the process of search for a proof, it is necessary to expand the notion of mathematical proof and introduce the concept of *proof-events* or *provings* (Goguen, 2001), which is described in more detail in the previous conference (Vandoulakis, 2016).

2. Problem Statement

Many researchers argue that the role of argumentation is so central to mathematics, as proof itself. Mathematicians do much more than just prove theorems. A significant part of their proving activity can be understood as argumentation (Aberdein, 2008). Lakatos's classic treatise *Proofs and Refutations* (Lakatos, 1976), highlights the role of dialogue between agents (teacher and pupils) in their attempt to prove a mathematical proposition. The comparison between an argument that justifies a hypothesis or a purported proof and the proof itself is based on the fact that the proof can be regarded as a special case of argumentation in mathematics (Pedemonte, 2007).

3. Research Questions

The methodology commonly used to study argumentation is based on Toulmin's model (Toulmin, 1958). Toulmin proposed a set of six interrelated components for the analysis of arguments:

- *Claim, conclusion*, i.e. the statement needed to be established.
- *Ground, fact, evidence, data*, i.e. the facts referred to as the basis of the claim.
- *Warrant*, i.e. a reasoning that allows one to move from the ground/ data to affirmation.
- *Backing*, i.e. additional reasoning aiming to confirm the statements expressed in the grounds. This is necessary when the grounds themselves are not sufficiently convincing.
- *Rebuttal*, i.e. a reasoning showing the possible conditional correctness of the grounds or their unsoundness.
- *Qualifier*. These are words and statements expressing the degree of the author's confidence in his statement. These are expressions of the type "probably", "possibly", "impossible", "unconditionally", "presumably", etc.).

The first three elements – "affirmation", "ground" and "warrant" – are considered as the main components of the argument, whereas the last three – "backing", "rebuttal" and "qualifier" – are not always necessary.

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4. Purpose of the Study

The aim of the research is to approach from a new perspective the question of the role played by argumentation in the process of searching for mathematical proof.

The procedure by which mathematicians evaluate reasoning resembles to argumentation, as many researchers have shown by adapting Toulmin's model to mathematical practice (Alcolea 1998; Aberdein 2005, 2008, 2009; Pedemonte 2007, 2008; Aberdein, Dove 2013). Thus, we proceed from a comparison of proof-events and argumentation, which serves as a basis for an integrated theory of proof-events and argumentation theories.

5. Research Methods

The current analysis is based on two major theories:

1. The theory of proof-events, initially exposed in (Stefaneas, Vandoulakis, 2014) and formalised in the calculus of proof-events (Stefaneas, Vandoulakis, 2015).
2. Argumentation theory, notably Toulmin's model (Toulmin, 1958) and Pollock's logical theory of argument (Pollock, 1987; 1992).

6. Findings

We claim that the combination of the theory of argumentation with the theory of proof-events enables us to represent all the steps in a process of searching for mathematical proof, starting from the statement of the problem to its acceptance or rejection by the relevant community participating in the proof-event, and identify the place and role of argumentation in this process.

Comparison of the main components of proof-events with the components of the arguments reveals certain structural similarities. Both arguments and proof-events share similar components:

a) The *claim* in the theory of argumentation corresponds to the *problem* posed in the theory of proof events. In the case of a proof confirmed by the community, the *derivation* corresponds to the *theorem*.
b) The *grounds* of a claim in the theory of argumentation correspond to the *axioms* and established *mathematical facts* in the theory of proof-events.
c) The *warrant* in the theory of argument corresponds to the *rules of inference* and *modes of reasoning* used by an agent enacting the role of prover when the sequence of proof-events develops in time.
d) *Backing* corresponds to sub-sequences of proof-events aimed at establishing auxiliary statements by agents enacting the role of prover.
e) *Rebuttal* (refutation, counter-arguments) corresponds to the application by agents enacting the role of prover or interpreter of logical means to verify the correctness or conditional validity of supposedly established statements.
f) To the *qualifier* correspond the estimations stated by the interpreters from the community, expressing the degree of their belief in the correctness of the output (conclusion).

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Like argumentation which assume at least two agents enacting the roles *supporter* and *opponent* of an argument (Kakas, Moraitis, 2003), proof-events also presuppose the existence of at least two agents enacting the roles of *prover* and *interpreter*.

The levels of communication, understanding, interpretation and verification used by agents to convey information are common in both approaches.

However, unlike argumentation, proof-events and their sequences have temporal extension and, thereby, history, i.e. incorporate the history of proof. Proof-events include also unproved and possibly even false statements, insights, intuitive ideas, etc. Moreover, the fact that proof-events generate proofs in different *styles* has no analogue in the theory of argumentation.

Nevertheless, the process of exchange of arguments and counterarguments, aimed at clarifying possible gaps or implicit assumptions that occur during the search for mathematical proof, can be formalized by resorting to argumentation theories.

By expanding the calculus of proof-events with the theory of argumentation we can take into consideration such moments as incomplete or even false purported proofs, intuitive ideas, correct or incorrect steps of reasoning, commentaries, etc. and represent them formally at the appropriate stages that take place during the evolution of a sequence of proof-events. This can be represented in the form of *dialogue* between agents who set forth arguments and counterarguments (or counterexamples) in their attempts to clarify the truth of a purported proof (Almpani, Stefaneas, Vandoulakis, 2017).

This expansion is possible by using the basic structure of Toulmin's model to represent arguments in combination with Pollock's logical theory of argument (Pollock 1987; 1992). In this way, the concept of *defeasible reasoning* can be used, i.e. the type of reasoning, when the corresponding argument is rationally convincing, although not deductive. The truth of the premises of a defeasible argument provides support for the inference, even if the premises are possible and the conclusion is false. In other words, the relation of support for the conclusion by the premises is probationary, potentially cancellable by additional information.

7. Conclusion

In this paper, a comparative analysis was carried out between the theory of proof-events and the theory of argumentation. The combination of these two theories enables us to represent controversial points in the process of searching for proof. Proof-events are not considered infallible facts, until they are finally confirmed and validated by the relevant community. This allows one to examine episodes of erroneous approaches and purported proofs using the theory of argument put forward by Toulmin (1958) and Pollock (1992) (as arguments and counterarguments) and the concept of defeasible reasoning. Thus, the exchange of arguments and counterarguments between agents, aimed at clarifying possible gaps or implicit assumptions that occur during a proof, can be formally represented within this integrated framework (Almpani, Stefaneas, Vandoulakis, 2017).

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