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The European Proceedings of
Social & Behavioural Sciences
EpSBS

eISSN: 2357-1330

**ERD 2016**: Education, Reflection, Development, Fourth Edition

# An Example of the Training a Methodological Competences in Mathematical Analysis

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**Abstract**

*http://dx.doi.org/10.15405/epsbs.2016.12.78*

In this paper we try to present some simple methods of training and development of professional competences, more specifically methodological, for obtaining the identities mathematical (combinatorial) and their use. Thus, starting from a simple integral, we obtain multiple identities mathematical (combinatorial), that, then we will use in calculating limits of sequences and some primitives, respectively definite integrals. The attentive reader and interested in these issues will notice that the results of this work complete and definitive resolves many problems of Mathematics. Therefore we consider that the work will be of real help to students who are preparing for competitions and Olympiads, students in deepening their knowledge and passing exams and teachers in their professional training. Of course not here we present all types of exercises that can solve with the formulas proven in the paper. There are other types of exercises than those presented herein and we will present in a forthcoming paper.



**Keywords:** Professional competences; identities mathematical; limits of sequences; primitives, definite integrals.

## 1. Introduction

As we have presented in Abstract, in this paper we intend to present some simple ways of training and development of professional competences, more specifically methodological, for obtaining the identities mathematical (combinatorial) and their use. Thus, starting from a simple integral, we obtain multiple identities mathematical (combinatorial), that, then we will use in calculating some limits of

*eISSN: 2357-1330*
*Selection and peer-review under responsibility of the Organizing Committee of the conference*

sequences and some primitives, respectively definite integrals. The attentive reader interested in these issues and will notice that the results of this work complete and definitive resolves many problems of Mathematics, using, exclusively, the results can be presented in class, so in preuniversity education, without using superior results in Mathematics.

## 2. Main results

The main results of the paper we will obtain starting from a definite integral. So, let be p, q∈**N** and consider the definite integral – see Mocica (1988) or http://mathworld.wolfram.com/zetafunction:

$$I(p.q)=\int_0^1 (1-x)^p \cdot x^q \cdot dx. \tag{1}$$

Using the binomial theorem (of Newton), we obtain that:

$$I(p.q)=\int_0^1 (1-x)^p \cdot x^q \cdot dx=\int_0^1 \sum_{k=0}^{p} (-1)^k \cdot C_p^k \cdot x^{q+k} \cdot dx=\sum_{k=0}^{p} (-1)^k \cdot C_p^k \cdot \frac{1}{q+k+1}. \tag{2}$$

But, by making change of variable 1-x=y, obtain that:

$$I(p.q)=-\int_1^0 y^p \cdot (1-y)^q \cdot dy=\int_0^1 (1-x)^q \cdot x^p \cdot dx = I(q,p). \tag{3}$$

By proceeding as above – in (2), obtain that:

$$I(q,p)=\sum_{k=0}^{q} (-1)^k \cdot C_q^k \cdot \frac{1}{p+k+1}. \tag{4}$$

From the equalities (2), (3) and (4) it follows that:

$$\sum_{k=0}^{p} (-1)^k \cdot C_p^k \cdot \frac{1}{q+k+1} = \sum_{k=0}^{q} (-1)^k \cdot C_q^k \cdot \frac{1}{p+k+1}; \tag{5}$$

i.e.:

$$\frac{C_p^0}{q+1}-\frac{C_p^1}{q+2}+\frac{C_p^3}{q+3}-\cdots+(-1)^p\cdot\frac{C_p^p}{q+p+1}=\frac{C_q^0}{p+1}-\frac{C_q^1}{p+2}+\frac{C_q^3}{p+3}-\cdots+(-1)^q\cdot\frac{C_q^q}{q+p+1}. \tag{5'}$$

On the other hand, integrating by parts, obtain that:

$$I(p.q)=\left.\frac{x^{q+1}\cdot(1-x)^p}{q+1}\right|_0^1+\frac{p}{q+1}\cdot\int_0^1 (1-x)^{p-1}\cdot x^{q+1}\cdot dx=\frac{p}{q+1}\cdot I(p-1,q+1). \tag{6}$$

From the equalities (6) it follows that:

$$I(p,q)=\frac{p}{q+1}\cdot I(p-1,q+1)=\cdots=\frac{p\cdot(p-1)\cdot(p-2)\cdots 2\cdot 1}{(q+1)\cdot(q+2)\cdot(q+3)\cdots(q+p-1)\cdot(q+p)}\cdot I(0,q+p)$$

$$=\frac{p\cdot(p-1)\cdot(p-2)\cdots 2\cdot 1}{(q+1)\cdot(q+2)\cdot(q+3)\cdots(q+p-1)\cdot(q+p)}\cdot\int_0^1 x^{q+p}\cdot dx=\frac{p!\cdot q!}{(q+p+1)!}. \tag{7}$$

From the equalities (3) and (7) it follows that:

$$I(p,q)=\frac{q!}{(p+1)\cdot(p+2)\cdot\dots\cdot(p+q)\cdot(p+q+1)}. \tag{8}$$

Now, from the equalities (7) and (8), obtain the equalities obvious: for every p, q∈**N**,

$$\frac{p!}{(q+1)\cdot(q+2)\cdot\dots\cdot(q+p)\cdot(q+p+1)}=\frac{q!}{(p+1)\cdot(p+2)\cdot\dots\cdot(p+q)\cdot(p+q+1)}=\frac{p!\cdot q!}{(q+p+1)!}. \tag{9}$$

On the other hand, from the equalities (2) and (7) it follows that:

$$\frac{1}{(q+1)\cdot\dots\cdot(q+p+1)}=\frac{1}{p!}\cdot I(p.q)=\frac{1}{p!}\cdot\sum_{k=0}^{p}(-1)^k\cdot C_p^k\cdot\frac{1}{q+k+1}=\frac{1}{p!}\cdot\sum_{k=0}^{q}(-1)^k\cdot C_q^k\cdot\frac{1}{p+k+1}. \tag{10}$$

Analogous, obtain that:

$$\frac{1}{(p+1)\cdot(p+2)\cdot\dots\cdot(p+q)\cdot(p+q+1)}=\frac{1}{q!}\cdot I(q,p)=\frac{1}{q!}\cdot\sum_{k=0}^{q}(-1)^k\cdot C_q^k\cdot\frac{1}{p+k+1}$$

$$=\frac{1}{q!}\cdot\left[\frac{C_q^0}{p+1}-\frac{C_q^1}{p+2}+\frac{C_q^2}{p+3}-\dots+(-1)^{q-1}\cdot\frac{C_q^{q-1}}{p+q}+(-1)^q\cdot\frac{C_q^q}{p+q+1}\right]$$

$$=\frac{1}{0!\cdot q!}\cdot\frac{1}{p+1}-\frac{1}{1!\cdot(q-1)!}\cdot\frac{1}{p+2}+\frac{1}{2!\cdot(q-2)!}\cdot\frac{1}{p+3}-\dots+(-1)^{q-1}\cdot\frac{1}{(q-1)!\cdot 1!}\cdot\frac{1}{p+q}+(-1)^{q}\cdot\frac{1}{q!\cdot 0!}\cdot\frac{1}{p+q+1}$$

$$=\frac{1}{q!}\cdot\sum_{k=0}^{p}(-1)^k\cdot C_p^k\cdot\frac{1}{q+k+1}. \tag{10′}$$

## 3. Consequences (1)

The theoretical results obtained above are entails two consequences important data in particular cases.

**1)** For every p, n∈**N**,

$$I(p,n)=\int_0^1(1-x)^p\cdot x^n\cdot dx=\frac{p!}{(n+1)\cdot\dots\cdot(n+p+1)}=\frac{p!\cdot n!}{(n+p+1)!}=\sum_{k=0}^{p}(-1)^k\cdot C_p^k\cdot\frac{1}{n+k+1}$$

$$=\sum_{k=0}^{n}(-1)^k\cdot C_n^k\cdot\frac{1}{p+k+1}=\frac{n!}{(p+1)\cdot\dots\cdot(p+n+1)}=\int_0^1(1-x)^n\cdot x^p\cdot dx=I(n,p). \tag{11}$$

Therefore,

$$\frac{1}{(n+1)\cdot(n+2)\cdot\dots\cdot(n+p)\cdot(n+p+1)}=\frac{1}{p!}\cdot I(p,n)=\frac{1}{p!}\cdot\sum_{k=0}^{p}(-1)^k\cdot C_p^k\cdot\frac{1}{n+k+1}$$

$$=\frac{1}{0!\cdot p!}\cdot\frac{1}{n+1}-\frac{1}{1!\cdot(p-1)!}\cdot\frac{1}{n+2}+\dots+(-1)^{p-1}\cdot\frac{1}{(p-1)!\cdot 1!}\cdot\frac{1}{n+p}+(-1)^{p}\cdot\frac{1}{p!\cdot 0!}\cdot\frac{1}{n+p+1}$$

and

$$\frac{1}{n!}\cdot\left[\frac{C_n^0}{p+1}-\frac{C_n^1}{p+2}+\frac{C_n^2}{p+3}-\cdots+(-1)^{n-1}\cdot\frac{C_n^{n-1}}{p+n}+(-1)^n\cdot\frac{C_n^n}{p+n+1}\right]$$

$$=\frac{1}{n!}\cdot\sum_{k=0}^{p}(-1)^k\cdot C_p^k\cdot\frac{1}{n+k+1}=\frac{1}{n!}\cdot I(n,p)=\frac{1}{(p+1)\cdot(p+2)\cdot\cdots\cdot(p+n)\cdot(p+n+1)}. \quad (11')$$

**2)** For every n∈**N**,

$$I(n,n)=\int_0^1(1-x)^n\cdot x^n\cdot dx=\frac{(n!)^2}{(n+n+1)!}=\frac{1}{(n+1)\cdot C_{2n+1}^n}=\sum_{k=0}^{n}(-1)^k\cdot C_n^k\cdot\frac{1}{n+k+1}. \quad (12)$$

## 4. Generalizations

Now we generalize the results from the first two paragraphs. Thus, let be p, q∈**N**, a, b∈**R**, with a<b and consider the definite integral:

$$J(p.q)=\int_a^b(b-x)^p\cdot(x-a)^q\cdot dx. \quad (13)$$

Using the binomial theorem (of Newton), we obtain that, for every p, q∈**N** and every a, b∈**R**, with a<b:

$$J(p.q)=\int_a^b(b-x)^p\cdot(x-a)^q\cdot dx=\int_a^b\left[\sum_{k=0}^{p}(-1)^k\cdot C_p^k\cdot b^{p-k}\cdot x^k\cdot\sum_{i=0}^{q}(-1)^i\cdot C_q^i\cdot x^{q-i}\cdot a^i\right]\cdot dx$$

$$=\int_0^1\sum_{k=0}^{p}\sum_{i=0}^{q}(-1)^{k+i}\cdot C_p^k\cdot C_q^i\cdot b^{p-k}\cdot a^i\cdot x^{q+k-i}\cdot dx$$

$$=\sum_{k=0}^{p}\sum_{i=0}^{q}(-1)^{k+i}\cdot C_p^k\cdot C_q^i\cdot b^{p-k}\cdot a^i\cdot\frac{b^{q+k-i+1}-a^{q+k-i+1}}{q+k-i+1}. \quad (14)$$

But, by making change of variable x=a+b-y, obtain that, for every p, q∈**N** and a, b∈**R**, with a<b:

$$J(p.q)=-\int_b^a(x-a)^p\cdot(b-x)^q\cdot dx=\int_a^b(b-x)^q\cdot(x-a)^p\cdot dx=J(q,p). \quad (15)$$

By proceeding as above, obtain that, for every p, q∈**N** and every a, b∈**R**, with a<b:

$$J(q,p)=\sum_{k=0}^{q}\sum_{i=0}^{p}(-1)^{k+i}\cdot C_q^k\cdot C_p^i\cdot b^{q-k}\cdot a^i\cdot\frac{b^{p+k-i+1}-a^{p+k-i+1}}{p+k-i+1}. \quad (16)$$

From the equalities (14), (15) and (16) it follows that, for every p, q∈**N** and every a, b∈**R**, with a<b:

$$\sum_{k=0}^{p}\sum_{i=0}^{q}(-1)^{k+i}\cdot C_p^k\cdot C_q^i\cdot b^{p-k}\cdot a^i\cdot\frac{b^{q+k-i+1}-a^{q+k-i+1}}{q+k-i+1}$$

$$=\sum_{k=0}^{q}\sum_{i=0}^{p}(-1)^{k+i}\cdot C_q^k\cdot C_p^i\cdot b^{q-k}\cdot a^i\cdot\frac{b^{p+k-i+1}-a^{p+k-i+1}}{p+k-i+1}. \quad (17)$$

On the other hand, integrating by parts, obtain that, for every p, q∈**N** and every a, b∈**R**, with a<b:

$$J(p.q)=\frac{(x-a)^{q+1}\cdot(b-x)^{p}}{q+1}\Bigg|_{a}^{b}+\frac{p}{q+1}\cdot\int_{a}^{b}(b-x)^{p-1}\cdot(x-a)^{q+1}\cdot dx=\frac{p}{q+1}\cdot J(p\text{-}1,q\text{+}1). \quad (18)$$

From the equalities (18) it follows that, for every p, q∈**N** and every a, b∈**R**, with a<b:

$$J(p,q)=\frac{p}{q+1}\cdot J(p\text{-}1,q\text{+}1)=\cdots=\frac{p\cdot(p-1)\cdot(p-2)\cdot\cdots\cdot 2\cdot 1}{(q+1)\cdot(q+2)\cdot(q+3)\cdot\cdots\cdot(q+p-1)\cdot(q+p)}\cdot J(0,q+p)$$

$$=\frac{p\cdot(p-1)\cdot(p-2)\cdot\cdots\cdot 2\cdot 1}{(q+1)\cdot(q+2)\cdot(q+3)\cdot\cdots\cdot(q+p-1)\cdot(q+p)}\cdot\int_{a}^{b}(x-a)^{q+p}\cdot dx$$

$$=\frac{p!}{(q+1)\cdot(q+2)\cdot\cdots\cdot(q+p)\cdot(q+p+1)}\cdot(b\text{-}a)^{q+p+1}=\frac{p!\cdot q!}{(q+p+1)!}\cdot(b\text{-}a)^{q+p+1}. \quad (19)$$

From the equalities (15) and (19) it follows that, for every p, q∈**N** and every a, b∈**R**, with a<b:

$$J(p,q)=\frac{q!}{(p+1)\cdot(p+2)\cdot\cdots\cdot(p+q)\cdot(p+q+1)}\cdot(b\text{-}a)^{q+p+1}. \quad (20)$$

Therefore, from the equalities (19) and (20), obtain again the equalities (9).

From the equalities (14) şi (19) it follows that, for every p, q∈**N** and every a, b∈**R**, with a<b:

$$\frac{1}{(q+1)\cdot(q+2)\cdot\cdots\cdot(q+p)\cdot(q+p+1)}=\frac{1}{(b-a)^{q+p+1}\cdot p!}\cdot J(p.q)$$

$$=\frac{1}{(b-a)^{q+p+1}\cdot p!}\cdot\sum_{k=0}^{p}\sum_{i=0}^{q}(-1)^{k+i}\cdot C_p^k\cdot C_q^i\cdot b^{p-k}\cdot a^i\cdot\frac{b^{q+k-i+1}-a^{q+k-i+1}}{q+k-i+1}. \quad (21)$$

Analogous, obtain that, for every p, q∈**N**:

$$\frac{1}{(p+1)\cdot(p+2)\cdot\cdots\cdot(p+q)\cdot(p+q+1)}=\frac{1}{(b-a)^{q+p+1}\cdot q!}\cdot J(q,p)=$$

$$=\frac{1}{(b-a)^{q+p+1}\cdot q!}\cdot\sum_{k=0}^{q}\sum_{i=0}^{p}(-1)^{k+i}\cdot C_q^k\cdot C_p^i\cdot b^{q-k}\cdot a^i\cdot\frac{b^{p+k-i+1}-a^{p+k-i+1}}{p+k-i+1}. \quad (21')$$

## 5. Consequences (2)

Now, we present two consequences of the results of the previous paragraph, obtained on particular cases.

**1)** For every p, n∈**N** and every a, b∈**R**, with a<b:

$$J(p,n)=\int_{a}^{b}(b-x)^{p}\cdot(x-a)^{n}\cdot dx=\frac{p!}{(n+1)\cdot\cdots\cdot(n+p+1)}\cdot(b\text{-}a)^{n+p+1}=\frac{p!\cdot n!}{(n+p+1)!}\cdot(b\text{-}a)^{n+p+1}$$

$$=\sum_{k=0}^{p}\sum_{i=0}^{n}(-1)^{k+i}\cdot C_p^k\cdot C_n^i\cdot b^{p-k}\cdot a^i\cdot\frac{b^{n+k-i+1}-a^{n+k-i+1}}{n+k-i+1}$$

$$=\sum_{k=0}^{n}\sum_{i=0}^{p}(-1)^{k+i}\cdot C_n^k\cdot C_p^i\cdot b^{n-k}\cdot a^i\cdot\frac{b^{p+k-i+1}-a^{p+k-i+1}}{p+k-i+1}$$

$$=\frac{n!}{(p+1)\cdot\dots\cdot(p+n+1)}\cdot(b-a)^{n+p+1}=\int_a^b (b-x)^n\cdot(x-a)^p\cdot dx=J(n,p). \quad (22)$$

Therefore,

$$\frac{1}{(n+1)\cdot(n+2)\cdot\dots\cdot(n+p)\cdot(n+p+1)}=\frac{1}{(b-a)^{n+p+1}\cdot p!}\cdot J(p,n)$$

$$=\frac{1}{(b-a)^{n+p+1}\cdot p!}\cdot\sum_{k=0}^{p}\sum_{i=0}^{n}(-1)^{k+i}\cdot C_p^k\cdot C_n^i\cdot b^{p-k}\cdot a^i\cdot\frac{b^{n+k-i+1}-a^{n+k-i+1}}{n+k-i+1}$$

$$=\frac{1}{(b-a)^{n+p+1}\cdot n!}\cdot\sum_{k=0}^{n}\sum_{i=0}^{p}(-1)^{k+i}\cdot C_n^k\cdot C_p^i\cdot b^{n-k}\cdot a^i\cdot\frac{b^{p+k-i+1}-a^{p+k-i+1}}{p+k-i+1}$$

$$=\frac{1}{(b-a)^{n+p+1}\cdot n!}\cdot J(n,p)=\frac{1}{(p+1)\cdot(p+2)\cdot\dots\cdot(p+n)\cdot(p+n+1)}. \quad (22')$$

**2)** For every $n\in\mathbf{N}$, and every $a, b\in\mathbf{R}$, with $a<b$:

$$J(n,n)=\int_a^b (b-x)^n\cdot(x-a)^n\cdot dx=\frac{n!}{(n+1)\cdot\dots\cdot(n+n+1)}\cdot(b-a)^{2n+1}=\frac{(n!)^2}{(n+n+1)!}\cdot(b-a)^{2n+1}$$

$$=\frac{1}{(n+1)\cdot C^1_{2\cdot n+1}}\cdot(b-a)^{2n+1}=\sum_{k=0}^{n}\sum_{i=0}^{n}(-1)^{k+i}\cdot C_n^k\cdot C_n^i\cdot b^{n-k}\cdot a^i\cdot\frac{b^{n+k-i+1}-a^{n+k-i+1}}{n+k-i+1}. \quad (23)$$

In particular,

**3)** For every $n\in\mathbf{N}$,

$$J(n,n)=\int_0^1 (1-x^2)^n\cdot dx=\frac{1}{2}\cdot\int_{-1}^{1}[(1-x)^n\cdot(1+x)^n]\cdot dx=\frac{1}{2}\cdot\frac{n!}{(n+1)\cdot\dots\cdot(n+n+1)}\cdot 2^{2n+1}$$

$$=\frac{(n!)^2}{(n+n+1)!}\cdot 2^{2n}=\frac{1}{(n+1)\cdot C^n_{2n+1}}\cdot 2^{2n}. \quad (24)$$

So, the sequences $(J(n,n))_{n\in\mathbf{N}}$ above is decreasing and has limits 0.

## 6. Applications

### *6.1 The calculation of some limits of sequences*

Using those presented in previous paragraphs we can calculate a series of sequences limits. Thus, we can show that the following equalities hold:

**1)** $\lim\limits_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+p)}=\frac{1}{p}\cdot\int_0^1\left[\frac{(1-x^p)\cdot x^{q-1}}{1-x^q}\right]\cdot dx$, $p, q\in\mathbf{N}$, $p<q$.

**2)** $\lim\limits_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(q\cdot k+p)\cdot(q\cdot k+r)}=\frac{1}{r-p}\cdot\int_0^1\left[\frac{(1-x^{r-p})\cdot x^{p-1}}{1-x^q}\right]\cdot dx$, $p, q, r\in\mathbf{N}$, $p<r<q$.

**3)** $\lim\limits_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+p)\cdot(q\cdot k+r)}=\frac{1}{p\cdot r\cdot(r-p)}\cdot\int_0^1\left[\frac{[(r-p)-r\cdot x^p+p\cdot x^r]\cdot x^{q-1}}{1-x^q}\right]\cdot dx$.

http://dx.doi.org/10.15405/epsbs.2016.12.78
eISSN: 2357-1330 / Corresponding Author: Teodor-Dumitru Vălcan
Selection and peer-review under responsibility of the Organizing Committee of the conference

**4)** $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{(q\cdot k+p)\cdot(q\cdot k+r)\cdot(q\cdot k+s)}=$

$\frac{1}{(r-p)\cdot(s-p)\cdot(s-r)}\cdot\int\limits_0^1\left[\frac{[(s-r)-(s-p)\cdot x^{r-p}+(r-p)\cdot x^{s-p}]\cdot x^{p-1}}{1-x^q}\right]\cdot dx$, p, q, r, s∈**N**, p<r<s<q.

**5)** $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{6}\cdot\int\limits_0^1\frac{(1-x)^3}{1-x^q}\cdot dx$.

**6)** $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)}=\frac{1}{6}\cdot\int\limits_0^1\left[\frac{x^{q-1}\cdot(1-x)^3}{1-x^q}\right]\cdot dx$.

**7)** $\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{(q\cdot k+1)\cdot(q\cdot k+2)\cdot\cdots\cdot(q\cdot k+p+1)}=\frac{1}{p!}\cdot\int\limits_0^1\frac{(1-x)^p}{1-x^q}\cdot dx$.

**8)** $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{(q\cdot k)\cdot(q\cdot k+1)\cdot(q\cdot k+2)\cdot\cdots\cdot(q\cdot k+p+1)}=\frac{1}{p!}\cdot\int\limits_0^1\left[\frac{x^{q-1}\cdot(1-x)^p}{1-x^q}\right]\cdot dx$.

To prove Exercise 5): For every k∈**N**, p=3, from the equalities (10) it follows that:

$$\frac{1}{(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\frac{1}{3!}\cdot\int\limits_0^1(1-x)^3\cdot x^{q\cdot k}\cdot dx$$

$$=\frac{1}{6}\cdot\left(\frac{1}{q\cdot k+1}-\frac{3}{q\cdot k+2}+\frac{3}{q\cdot k+3}-\frac{1}{q\cdot k+4}\right)=\frac{1}{6}\cdot\int\limits_0^1(x^{q\cdot k}-3\cdot x^{q\cdot k+1}+3\cdot x^{q\cdot k+2}-3\cdot x^{q\cdot k+3})\cdot dx.$$

So,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(q\cdot k+1)\cdot(q\cdot k+2)\cdot(q\cdot k+3)\cdot(q\cdot k+4)}=\lim_{n\to\infty}\frac{1}{6}\cdot\sum_{k=0}^{n}\int\limits_0^1(1-x)^3\cdot x^{q\cdot k}\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int\limits_0^1\sum_{k=0}^{n}(1-x)^3\cdot x^{q\cdot k}\cdot dx=\frac{1}{6}\cdot\lim_{n\to\infty}\int\limits_0^1(1-x)^3\cdot\sum_{k=0}^{n}x^{q\cdot k}\cdot dx$$

$$=\frac{1}{6}\cdot\lim_{n\to\infty}\int\limits_0^1(1-x)^3\cdot\frac{1-x^{q\cdot n+q}}{1-x^q}\cdot dx=\frac{1}{6}\cdot\int\limits_0^1\frac{(1-x)^3}{1-x^q}\cdot dx.$$

In particular, for q=3:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(3\cdot k+1)\cdot(3\cdot k+2)\cdot(3\cdot k+3)\cdot(3\cdot k+4)}=\frac{1}{3!}\cdot\lim_{n\to\infty}\sum_{k=0}^{n}\int\limits_0^1(1-x)^3\cdot x^{3\cdot k}\cdot dx$$

$$=\frac{1}{3!}\cdot\int\limits_0^1\frac{(1-x)^2}{1+x+x^2}\cdot dx=\frac{1}{3!}\cdot\int\limits_0^1\left(1-\frac{3}{2}\cdot\frac{2\cdot x+1}{x^2+x+1}+\frac{3}{2}\cdot\frac{1}{x^2+x+1}\right)\cdot dx=\frac{1}{6}-\frac{1}{4}\cdot\ln3+\frac{\pi\cdot\sqrt{3}}{36}.$$

For the calculation of the following limits of sequences we need the following technical result:

***Proposition***: *For every continuous function f : [0,1] → (0,+∞),*

$$\lim_{n\to\infty}\int\limits_0^1[f(x)\cdot x^n]\cdot dx=0. \qquad (25)$$

***Proof***: Indeed, if we denote with $I_n=\int_0^1 [f(x)\cdot x^n]\cdot dx$, then, is checked immediately that the sequence $(I_n)_{n\in N}$ is decreasing and bordered – i.e., for every n∈**N**, $I_n\in[0,I_0]$, where $I_0=\int_0^1 f(x)\cdot dx$. Therefore, the sequence $(I_n)_{n\in N}$ is convergent. We notice also that, for every n∈**N**, $I_n\leq\frac{M}{n+1}$, where $M=\max_{x\in[0,1]} f(x)$. Whence, for every n∈**N**,

$$0\leq I_n\leq\frac{M}{n+1}. \qquad (26)$$

Passing to limit in the inequalities (26), obtain the equality (25).

*6.2 The calculation of some primitives and definite integrals*

Returning to those presented in the first two paragraphs, observe that, for every p∈**N** and every x∈**R**\{-p-1,-p-2,⋯,-2,-1}, we have the equalities:

$$\frac{1}{p!}\cdot I(p,x)=\frac{1}{p!}\cdot\int_0^1(1-t)^p\cdot t^x\cdot dt=\frac{1}{(x+1)\cdot(x+2)\cdot\cdots\cdot(x+p+1)}=\frac{1}{p!}\cdot\sum_{k=0}^{p}(-1)^k\cdot C_p^k\cdot\frac{1}{x+k+1}$$

$$=\frac{1}{0!\cdot p!}\cdot\frac{1}{x+1}-\frac{1}{1!\cdot(p-1)!}\cdot\frac{1}{x+2}+-\cdots+(-1)^{n-1}\cdot\frac{1}{(p-1)!\cdot 1!}\cdot\frac{1}{x+n}+(-1)^{n}\cdot\frac{1}{p!\cdot 0!}\cdot\frac{1}{x+n+1}. \qquad (27)$$

Hence,

$$\int\frac{dx}{(x+1)\cdot(x+2)\cdot\cdots\cdot(x+p)\cdot(x+p+1)}=\frac{1}{p!}\cdot\sum_{k=0}^{p}(-1)^k\cdot C_p^k\cdot\int\frac{dx}{x+k+1}$$

$$=\frac{1}{p!}\cdot\sum_{k=0}^{p}(-1)^k\cdot C_p^k\cdot\ln(x+k+1)+C; \qquad (28)$$

and:

$$\int_a^b\frac{dx}{(x+1)\cdot(x+2)\cdot\cdots\cdot(x+p)\cdot(x+p+1)}=\frac{1}{p!}\cdot\sum_{k=0}^{p}(-1)^k\cdot C_p^k\cdot\int_a^b\frac{dx}{x+k+1}$$

$$=\frac{1}{p!}\cdot\sum_{k=0}^{p}(-1)^k\cdot C_p^k\cdot\ln\frac{b+k+1}{a+k+1}=\frac{1}{p!}\cdot\sum_{k=0}^{p}\ln\left(\frac{b+k+1}{a+k+1}\right)^{(-1)^k\cdot C_p^k}=\ln\sqrt[p!]{\prod_{k=0}^{p}\left(\frac{b+k+1}{a+k+1}\right)^{(-1)^k\cdot C_p^k}}. \qquad (29)$$

In particular, for p=3:

$$\int\frac{dx}{(x+1)\cdot(x+2)\cdot(x+3)\cdot(x+4)}=\frac{1}{3!}\cdot\sum_{k=0}^{3}(-1)^k\cdot C_3^k\cdot\int\frac{dx}{x+k+1}=\frac{1}{6}\cdot\sum_{k=0}^{3}(-1)^k\cdot C_3^k\cdot\ln(x+k+1)+C$$

and:

$$\int_2^5\frac{dx}{(x+1)\cdot(x+2)\cdot(x+3)\cdot(x+4)}=\frac{1}{3!}\cdot\sum_{k=0}^{3}(-1)^k\cdot C_p^k\cdot\int_2^5\frac{dx}{x+k+1}=\ln\sqrt[6]{\frac{131072}{128625}}.$$

So, the sequences $(J(n,n))_{n\in\mathbf{N}}$ above is decreasing and has limits 0.

*http://dx.doi.org/10.15405/epsbs.2016.12.78*
*eISSN: 2357-1330 / Corresponding Author: Teodor-Dumitru Vălcan*
*Selection and peer-review under responsibility of the Organizing Committee of the conference*

## 7. Conclusions

So from integral I (p, q), which can be calculated in two ways, we can get all identities from (2) to (10′). Particularizing these identities obtain the identities (11) and (11′). By passing from interval [0,1] at a certain interval [a, b], all identities (2) - (10′) can be generalized to yield the identities (13) to (21′) identities that then can be customized, achieving equalities (22) - (24), obtaining the identities from (13) to (21′), identities that can be particularized, obtaining the equalities (22)-(24). Finally, we presented how you can apply to those shown in the paper in calculating the limits of eight types of convergent sequences, of some primitive and definite integrals. For other examples see Sirețchi (1985).

## References

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