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Alpha Proportionality and Penrose Square Root Law

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Abstract

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Problem concerns the distribution of mandates in collegial bodies of communities of states. The number of delegates representing each member of community depends on the number of its population. The paper analyzes the fairness of different kinds of relationship involving the number of representatives in collegial bodies of communities of states, in search of the fairest relationship. The research aims at comparing different ideas of how the numbers of representatives are determined reflecting the population of member states. Literature review, elementary probabilistic tools and mathematical analysis of degressive proportionality are employed. The main result of the paper demonstrates that the two fundamentally different approaches to the problem of distribution of mandates in collegial bodies may lead to similar solutions. A probabilistic approach (based on the voting power) and the concept of degressive proportionality (without additional assumptions) do not exclude each other, but, quite the contrary, may complement each other. The Penrose law and the rule of degressive proportionality, when suitably interpreted, are two sides of the coin. Degressive proportionality, as written in the Lisbon Treaty, is a vague idea.

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1. **Introduction**

The influence of individual members of an organization, economic or political communities on decision making is one of the key issues in modern societies. A rich literature underlies the analyses of how to influence the decisions or voting results, especially as regards indirect voting. Advanced tools of game theory are frequently applied here. Felsenthal & Machover (1998), among others, comprehensively reviewed these issues. Lionel Penrose (1946) provided mathematical foundations of such analyses in the 1950s.

We focus in this paper on one practical case where Penrose's concept can be employed, i.e. the distribution of seats in the European Parliament among the member states of the European Union,

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considering the allocation source under the Lisbon Treaty – the rule of degressive proportionality (DP). However, our reflections are relatively general and can be easily employed in other cases of decision making by indirect voting.

The ability of a member of a community (an organization) to influence decisions taken is referred to as voting power. Voting weight denotes a potential influence of a community member on their decisions. The number of population in a member state of a community can be employed as a voting weight.

2. Penrose square root law

Let us consider a voting body consisting of *N* persons with identical voting rights. This is the case of parliamentary elections and also of a general meeting of shareholders. Let us assume that voters make a choice between "yes" and "no". The equality of voting rights implies that each vote has the same value. Assume further that a standard majority rule holds in this case. Because the weights of all votes are equal, the voting powers should also be equal. A question is significant, whether (how) an elector's voting power depends on the number *N*. The answer is essential in our subsequent analysis of voting powers of individual people in a given community of states. If voting power depends on *N*, the citizens of different member states have different potential voting powers. This means that the citizens of a community have unequal influence on decision making in their community. Let us consider the following example that demonstrates how a voting weight can be significantly different from a voting power.

An example

A community comprises three states *A*, *B* and *C*, with respective populations $a = 3$ and $b = c = 1$. A certain issue is to be decided upon in this community. The decision will be taken by representatives of member states, with weights assigned to them proportionally to the numbers of populations. Thus the weights of the states *A*, *B* and *C* equal 3, 1, 1, respectively. The representatives who make decisions (vote) in the name of their states are also elected by a standard majority rule. Thus a decision of each state is consistent with the attitude of a majority of its citizens. In such a case, the decision will always be made by the state *A*. The decision of the community will always be equal to that of *A*. Let us evaluate the voting powers of citizens in this community of states depending on their citizenship. Let us note that any two persons from the state *A* have a voting power. These two persons decide on who will represent them, and their representative has a voting weight equal to 3, while combined voting powers of the states *B* and *C* are equal 2. But two persons from *B* and *C* do not have such power. As a result, the influence of individual persons in the community depends on their citizenship. To sum up, the state A has 60 percent of the community population (voting weight) and 100 percent of voting power. What are the weights of the states *A*, *B* and *C* required so that all their citizens have equal rights? Definitely, the respective weights should be smaller than those resulting from the simple proportionality.

The example above demonstrates that proportionality, in spite of intuitively being the fairest solution in many social decisions, might actually be unfair. This is a result of indirect voting. The example is

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extremely unrealistic due to populations of the states *A*, *B* and *C*, but the problem illustrated is real. Mathematical theory of indirect voting and related issues was originated by Lionel Penrose (1946) who introduced a concept of voting power. He employed the distribution of votes in the United Nations General Assembly as a starting point. He assumed that the voting power of a given country is proportional to the probability that the vote by this country will be conclusive in a hypothetical voting. If we define a voting power in this way, *B*'s and *C*'s voting powers in our example are null.

We proceed now to reflections leading to the Penrose square root law. Earlier we asked a question whether, and how, a voting power depends on the population of country whose citizen is the owner of a vote. For simplicity, we assume that the population is an odd number, $N = 2k + 1$. Penrose argued that a voter will actually influence the voting result when remaining votes are distributed evenly. With k votes in favour of "yes" and k votes in favour of "no", our voter will effectively make a decision. Do we often deal with this type of situation? Without any information about the preferences of individual voters, we can assume that their voting decisions are independent. This is reflected mathematically by the Bernoulli scheme. Let P_n denote probability that n voters selects the option "yes". Then

$$P_n = \binom{2k}{n} p^n (1-p)^{2k-n}, \qquad 1)$$

where p denotes probability that a voter selects "yes". The lack of information about the preferences of voters allows to assume also that a vote "yes" is as likely as a vote "no", i.e. that $p = \frac{1}{2}$. Letting $p = \frac{1}{2}$ and $n = k$ in (1), as votes for and against are evenly distributed, gives

$$P_k = \binom{2k}{k} \left(\frac{1}{2}\right)^{2k} = \frac{(2k)!}{(k!)^2} \left(\frac{1}{2}\right)^{2k}. \qquad (2)$$

For large *N* we can obtain an approximate value of (2) applying Stirling approximation

$$k! \approx \sqrt{2\pi k} \left(\frac{k}{e}\right)^k, \qquad (3)$$

where e is Euler's number.

Then we obtain the probability that the vote of a given voter is conclusive:

$$\psi = P_k \approx 2^{-2k} \frac{\left(\frac{2k}{e}\right)^{2k} \sqrt{4\pi k}}{\left[\left(\frac{k}{e}\right)^k \sqrt{2\pi k}\right]^2} = \frac{1}{\sqrt{\pi k}} \approx \sqrt{\frac{2}{\pi N}}\,. \qquad (4)$$

For even N the result is the same as in (4). Thus probability that the vote of a given member of a voting body is conclusive, i.e. his/her voting power, depends on the frequency *N* of this body and is proportional to $1/\sqrt{N}$. This fact is called the Penrose square root law. The result holds under assumption that there is no correlation between decisions of respective electors. This assumption is crucial for the Penrose law, and is fully justified under limited information about potential correlated behaviours of electors[1]. Thus, if a voting power of a citizen from the country with population N is proportional to $1/\sqrt{N}$, then in case of a community of member states, a voting power of a country must be proportional to $\sqrt{N}$.

It should also be noted that actually the European Parliament is not quite appropriate forum where the Penrose law is to be applied. This results from the fact that formally the EP is an organ that represents citizens of the community, not its member states[2]. Nevertheless, as long as the composition of the EP is determined by elections held in the members states, not by selection of candidates proposed in so-called transnational lists, the representation of a country should be validly treated as its vote weighted by the number of allocated seats in the Parliament. This regards particularly the cases of voting where the interests of individual member states are crucial.

3. Degressive proportionality – between equality and proportionality

The term 'degressive proportionality' appears in one of the most important legal acts of the European Union – the Lisbon Treaty (LT). The LT stipulates that the allocation of seats in the EP among member states of the EU should be degressively proportional. The Treaty itself did nor precisely define the term and ordered the Parliament to accomplish the task in an appropriate resolution. The AFCO report (Lamassoure & Severin, 2007) reads that "…the ratio between the population and the number of seats of each Member State must vary in relation to their respective populations in such a way that each Member from a more populous Member State represents more citizens than each Member from a less populous Member State and conversely, but also that no less populous Member State has more seats than a more populous Member State".

If we assume that the two member states have their populations p_i and p_j with numbers of seats in the EP s_i and s_j, respectively, then degressive proportionality implies that the inequality $\frac{s_i}{p_i} < \frac{s_j}{p_j}$ holds always when $p_i > p_j$[3]. The Lamassoure and Severin report did not add anything new to the rule of degressive proportionality. It only introduced a general interpretation of the rule. From the very beginning of the EP, the rule (even if formally nonexistent in legal acts) was understood precisely as written by the authors of the report. A detailed analysis shows that the composition of the EP before 1995 was always degressively proportional, although legally the rule was never imposed. The violation of the rule occurred for the first time in 1995 after the accession of Austria, Finland and Sweden[4]. In the end, the composition of the Parliament elected for the 2009–2014 term did not comply with the DP rule. This was a consequence of delays in the ratification of the Lisbon Treaty by several member states that resulted in a suspension of the practical application of the DP rule.

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[1] There are many papers, e.g. Gelman et al. (2004), showing that often the assumptions of the Penrose law are not satisfied.
[2] The European Council represents the member states of the European Union.
[3] Degressive proportionality also requires that a less populous member state cannot have more seats than a more populous state. But this requirement is not challenged.
[4] Oddly enough, almost exactly at the same time when first attempts were launched to more precisely define the idea of degressive proportionality, the composition of the EP failed to satisfy the DP rule.

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A hypothetical composition of the EP elected according to the Treaty provisions is given in Martinez–Aroza & Ramirez (2008).

We proceed now to mathematical aspects of the DP. A degressiveness of a distribution implies that a big one gets more than a small one, but less than would be allocated with a proportional distribution. Hence, this is an intermediate solution between equality and proportionality. Let $f(p)$ denote the number of seats allocated to a country with population p. Degressive proportionality implies that the function $f(p)$ is nondecreasing and the function $f(p)/p$ is decreasing. Thus the two inequalities must be satisfied:

$$\frac{df(p)}{dp} \geq 0 \text{ and } \frac{df(p)}{dp} \leq \frac{f(p)}{p}, \qquad (5)$$

that give:

$$\frac{df(p)}{dp} \in \left[0, \frac{f(p)}{p}\right]. \qquad (6)$$

The relationship (6) is a formal mathematical representation of the idea of degressive proportionality, demonstrating in what sense the DP rule is an intermediate solution between equality and proportionality. The left endpoint of the interval in (6) yields an equal distribution – a derivative of the function $f(p)$ equals zero, and so each state, regardless of its population, is allocated the same number of seats. The righ endpoint of the interval in (6) results in $\frac{df(p)}{dp} = f(p)/p$. The only functions that satisfy this condition are linear functions, yielding a strictly proportional distribution.

4. Alpha proportionality

Problems emerging as regards the practical applications of degressive proportionality are caused by an ambiguity of the rule. Proportional divisions that are based on equality (unambiguity) are very intuitive, however there are also some troubles in practice, concerning the distribution of indivisible goods. Still, we know exactly what is expected from such distribution. The troubles and paradoxes encountered in proportional allocations are well known and represented in literature.

In case of degressive proportionality, in addition to rounding problems (indivisible goods), we also deal with another, much more significant, dilemma. Generally, there are a great number of allocations that satisfy the DP conditions. Łyko & Rudek (2013) report there are around 29 million allocations of seats considering the current populations of the EU member states. A choice of a given allocation must be done in at least two stages. For instance, first, we select a class of allocation functions, and then, its parameters are determined yielding an actual distribution. In case of the European Parliament, the experts strongly disagree as regards the selection of a class of allocation functions.

There are many classes of allocation functions (or algorithms) recommended by literature that can help compose the EP. One of the concepts that seems to be mostly preferred by member states is the allocation based on the so-called shifted proportionality (Pukelsheim, 2010). The idea was examined and developed by a team of mathematicians and politicians (Grimmett et al., 2011) and is known as the

Cambridge Compromise. A profound mathematical analysis of the degressive proportionality rule was conducted by Słomczyński & Życzkowski (2012).

Alpha proportionality is one of the propositions satisfying the DP conditions that are put forward by literature. The concept is based on a power function of allocation $f(p) = \beta p^{\alpha}$, with $\beta > 0$ and $\alpha \in [0,1]$. The allocation is conducted after adjusting the parameters α and β so that the so-called boundary conditions are fulfilled[5].

We present now a natural justification of the concept of alpha proportionality and observe how this concept leads to conclusions resulting from the Penrose law.

We already know that degressive proportionality at the level of allocation function implies that the relationship (6) holds. The nearer $\frac{df(p)}{dp}$ to the left endpoint of the interval $\left[0, \frac{f(p)}{p}\right]$, the closer the allocation obtained is to an equal distribution. Respectively, when shifting towards the right endpoint, we approach a proportional distribution. With no initial conditions imposed on $\frac{df(p)}{dp}$, one may state: let us adopt an intermediate solution. In other words, if we know only that (6) must be satisfied and nothing else, then let $\frac{df(p)}{dp}$ be the center of the interval $\left[0, \frac{f(p)}{p}\right]$, or

$$\frac{df(p)}{dp} = \frac{1}{2}\frac{f(p)}{p} \qquad (7)$$

5. Conclusion

The Penrose square root is strongly based on strict mathematical thinking and at the same time, the conclusions derived from the possibility of practical applications are remarkably simple and strongly inspiring. The Penrose method was a serious alternative against the solutions adopted by the Treaty of Nice as regards, among others, decision-making rules in the Council of the European Union. Słomczyński & Życzkowski (2010) proposed a root method (the voting power of each country is proportional to the square root of the population represented) that gained a significant support among member states. The Penrose law is a consequence of adopting some neutral and natural assumptions as regards the nature of the phenomenon under study. We mean the assumptions, firstly, that the decisions made by individuals are independent of each other, and secondly, that the probability of a decision in favour of "yes" is equal to the probability of a decision in favour of "no". When analyzing the idea of DP, we adopted such a refinement. If a degressively proportional distribution is supposed to be a compromise between equality and proportionality, but we do not know how far this compromise should go and towards which solution, then let us select an intermediate solution that is understood as it should be. It turns out that the two different approaches lead to identical solutions.

[5] Boundary conditions represent the requirements stipulated by the Lisbon Treaty as regards the minimum (6), maximum (96) and total (751) number of Members of the EP. Łyko (2012) and Łyko (2013), among others, examine the implications of boundary conditions in degressively proportional distributions.

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