European Proceedings of
Educational Sciences
EpES
www.europeanproceedings.com
e-ISSN: 2672-815X

DOI: 10.15405/epes.22032.63

ERD 2021
9th International Conference Education, Reflection, Development

FROM DIOFANTIAN EQUATIONS TO MATRICIAL EQUATIONS (II) -GENERALIZATIONS OF THE PYTHAGOREAN EQUATION-

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Abstract

In this paper we propose to continue the steps started in a first paper with the same generic title and marked symbolically with (I), namely, the presentation of ways to achieve a systemic vision on a certain notional mathematical content, a vision that motivates and mobilizes the activity of those who teach in the classroom, thus facilitating both the teaching and the assimilation of the notions, concepts, scientific theories approached by the educational disciplines that present phenomena and processes in nature. Thus, we will continue in the same systemic approach, solving some Diophantine equations, more precisely some generalizations of the Pythagorean equation and some quadratic Diophantine equations, in the set of natural numbers, then of integers and then "submerged" such an equation in a ring of matrices and try to find as many matrices solutions as possible. This paper has two large paragraphs. In the first paragraph we will generalize the known Pythagorean equation, in two forms, we will solve them in the set of natural numbers and then in Z, after which we will immerse them, both in the ring Mn(Z). In the second paragraph, we will proceed analogously with (other) four types of quadratic Diophantine equations. This paper is one of Didactics of Mathematics and is addressed to pupils, students or teachers attentive and interested in these issues, which we believe we have formed, in this way, a good image about solving these two types of equations.

Keywords: Equation, ring, matrix, system, solution

https://doi.org/10.15405/epes.22032.63
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Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2672-815X

1. Introduction

As mentioned above, in this paper we will continue the steps started in (Vălcan, 2019), namely, we will present a way to achieve a systemic vision on a certain notional mathematical content, more precisely, the transition from certain Diophantine equations to various matrix equations, a vision that motivates and mobilizes the activity of those who teach in the classroom, thus facilitating both teaching and assimilation of notions, concepts, scientific theories approached by educational disciplines that present phenomena and processes in nature.

Thus, we will continue in the same systemic approach, to solve some Diophantine equations, more precisely some generalizations of the Pythagorean equation and some quadratic Diophantine equations, in the set of natural numbers, then of integers and then "*submerged*" such an equation in a ring of matrices and try to find as many matriceal solutions as possible.

Therefore, this paper is a continuation of the paper (Vălcan, 2019). In this sense, we kept and continued not only the ideas but also the numbering of the paragraphs and the results.

This paper has two large paragraphs. In the first paragraph we will generalize the Pythagorean equation (A), in the form of equation (B) and in the form of equation (B″), we will solve them in the set of natural numbers and then in Z, after which we will immerse them, both in the ring Mn(Z). In the second paragraph, we will proceed analogously with (other) four types of quadratic Diophantine equations denoted by (C), (D) - with the particular cases (D″′) and (D(iv)), (E) and (F), respectively.

In solving these equations we will use didactic methods, easily accessible to pupils and students, different from those presented in (Andreescu & Andrica, 2002) or (Cucurezeanu, 2005), but based on ideas from there.

Also, here we specify that we will use the knowledge presented in (Acu, 2010), regarding the divisibility of integers.

2. Problem Statement

Solving Diophantine equations often proves to be quite difficult for pupils, students or teachers. In the pre-university curriculum this topic does not appear explicitly, that is why teachers do not allocate special lessons to teach methods to solve these types of equations. On the other hand, the number of these methods is quite large and such equations appear, since middle school, in solving divisibility problems - see (Vălcan, 2017).

This lack is found even further - in higher education, in the sense that, unfortunately, not all faculties of Mathematics have special courses aimed at solving Diophantine equations. So, it is possible that a graduate of such a faculty, who has become a professor, does not know much about Diophantine equations. Under such conditions he will not be able to teach his students how to solve such equations. We are not even talking about immersing such equations in various rings.

Therefore, this paper comes to reduce this shortcoming for both pupils and students, as well as for teachers.

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3. Research Questions

In our research we will try to find answers to the following questions:

- Can the Pythagorean equation be generalized?
- They have generalized Pythagorean equations solutions in the set of integers?
- If we immerse these generalized Pythagorean equations in the matrix ring, $M_n(\mathbf{Z})$, do the new matrix equations have solutions in this ring?

4. Purpose of the Study

Therefore, we will generalize, in several ways, the known Pythagorean equation:

$$x^2+y^2=z^2, \tag{A}$$

we will solve in the set **Z** all these generalized Pythagorean equations and we will study their solvability in a ring of matrices.

5. Research Methods

5.1. Generalizations of the Pythagorean equation

Immediate generalization of the Pythagorean equation (A) is given by the equation:

$$x^2+y^2+z^2=t^2. \tag{B}$$

The positive solutions of equation (B) represent the dimensions and the length of the diagonals of a rectangular parallelepiped. (Andreescu & Andrica, 2002) As in the case of the right triangle, here too we are interested in the situation in which all these dimensions are natural numbers.

For a start, three observations are required here as well:

Remark 5.1.1: *If the quadruple* $(x_0,y_0,z_0,t_0)\in\mathbf{N}\times\mathbf{N}\times\mathbf{N}\times\mathbf{N}$ *satisfies equation (B), then any quadruple of the form* $(k\cdot x_0,k\cdot y_0,k\cdot z_0,k\cdot t_0)$*, with* $k\in\mathbf{Z}$*, is a solution of this in* $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$*. Therefore, to solve equation (B) it is sufficient to determine the solutions* $(x,y,z,t)\in\mathbf{N}\times\mathbf{N}\times\mathbf{N}\times\mathbf{N}$ *with the property that*:

$$(x,y,z,t)=1. \tag{5.1.1}$$

That is why the following definition is required (and here):

Definition 5.1.2: *A solution* $(x_0,y_0,z_0,t_0)\in\mathbf{N}^*\times\mathbf{N}^*\times\mathbf{N}^*\times\mathbf{N}^*$ *of equation (B), which verifies the equality (5.1.1) is called a primitive solution.*

Remark 5.1.3: *Quadruples (0,y,z,t), (x,0,z,t) and (x,y,0,t), respectively, with x, y, z and t natural numbers, which reduce equation (B) to an equation of type (A) will not be analyzed in this paper; they can be seen in (Vălcan, 2019).*

Therefore, in the following, we will consider only solutions with non-zero components.

Remark 5.1.4: *Equation (B) is symmetric in x, y and z; so, if* $(x,y,z,t)\in\mathbf{N}^*\times\mathbf{N}^*\times\mathbf{N}^*\times\mathbf{N}^*$ *is its solution, then also (x,z,y,t), (y,x,z,t), (y,z,x,t), (z,x,y,t), (z,y,x,t)* $\in\mathbf{N}^*\times\mathbf{N}^*\times\mathbf{N}^*\times\mathbf{N}^*$ *are (also) solutions of the equation (B).*

We present below the form of a primitive solution of equation (B):

Theorem 5.1.5: *All primitive solutions in natural numbers of equation (B) are given by the equalities*:

$$x=2\cdot a, \qquad y=2b, \qquad z=\frac{a^2+b^2-c^2}{c} \text{ and } \qquad t=\frac{a^2+b^2+c^2}{c}, \tag{5.1.2}$$

where a, $b\in\mathbf{N}^*$*, and c is a divisor of* a^2+b^2*, which is less than* $\sqrt{a^2+b^2}$ *, and*:

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(a,b,c)=1. (5.1.1′)

Any solution is obtained only once in this way.

Proof: Let $(x,y,z,t)\in\mathbf{N}^*\times\mathbf{N}^*\times\mathbf{N}^*\times\mathbf{N}^*$ be a solution of equation (B). Based on Remark 2.4 from (Vălcan, 2019), we distinguish the following cases:

Case 1: All numbers x, y and z are odd. Then t will be odd; so:

$$x=2\cdot a+1,\quad y=2\cdot b+1,\quad z=2\cdot c+1\quad \text{and}\quad t=2\cdot d+1, \tag{5.1.3}$$

with a, b, c, $d\in\mathbf{N}^*$. From equalities (B) and (5.1.3) it follows that:

$$4\cdot a^2+4\cdot a+1+4\cdot b^2+4\cdot b+1+4\cdot c^2+4\cdot c+1=4\cdot d^2+4\cdot d+1, \tag{5.1.4}$$

that is:

$$4\cdot a^2+4\cdot a+4\cdot b^2+4\cdot b+4\cdot c^2+4\cdot c+2=4\cdot d^2+4\cdot d, \tag{5.1.4′}$$

which is impossible modulo 4; or otherwise: by dividing by 2, the left limb is an odd number and the right limb is an even number.

Case 2: One of the numbers x, y, z is even; say the number x is even, and the numbers y and z are odd. Then the number t is even:

$$x=2\cdot a,\quad y=2\cdot b+1,\quad z=2\cdot c+1\quad \text{and}\quad t=2\cdot d, \tag{5.1.5}$$

with a, b, c, $d\in\mathbf{N}^*$. From equalities (B) and (5.1.5) it follows that:

$$4\cdot a^2+4\cdot b^2+4\cdot b+1+4\cdot c^2+4\cdot c+1=4\cdot d^2, \tag{5.1.6}$$

that is:

$$4\cdot a^2+4\cdot b^2+4\cdot b+4\cdot c^2+4\cdot c+2=4\cdot d^2, \tag{5.1.6′}$$

which is impossible modulo 4; or otherwise: by dividing the equality (5.1.6′) by 2, the left limb is an odd number and the right limb is an even number.

Case 3: Two of the numbers x, y, z are even; say the numbers x and y are even and the number z is odd. Then the number t is odd, too. So:

$$x=2\cdot a,\quad \text{and}\quad y=2\cdot b, \tag{5.1.7}$$

with a, $b\in\mathbf{N}^*$. From equalities (B) and (5.1.7) it follows that:

$$4\cdot a^2+4\cdot b^2+z^2=t^2. \tag{5.1.8}$$

We note:

$$t-z=d. \tag{5.1.9}$$

Then, from equalities (B), (5.1.8) and (5.1.9), we obtain that:

$$4\cdot a^2+4\cdot b^2+z^2=(z+d)^2, \tag{5.1.8′}$$

whence it follows that:

$$4\cdot a^2+4\cdot b^2-d^2=2\cdot z\cdot d. \tag{5.1.8″}$$

Equality (5.1.8″) shows us that d is even, so:

$$d=2\cdot c, \tag{5.1.9}$$

with $d\in\mathbf{N}^*$. Now, from equalities (5.1.8″) and (5.1.9) it follows that:

$$z=\frac{a^2+b^2-c^2}{c}. \tag{5.1.10}$$

Finally, from equalities (5.1.9) and (5.1.10), it follows that:

$$t=\frac{a^2+b^2+c^2}{c}. \tag{5.1.11}$$

Because z, $t\in\mathbf{N}^*$, from any of the equalities (5.1.10) or (5.1.11), it follows that c is a divisor of the number a^2+b^2, and from the equality (5.1.10) it follows that $c<\sqrt{a^2+b^2}$.

Case 4: All numbers x, y and z are even. Then the number t is even and, thus, we contradict the equality (5.1.1), which we considered as a hypothesis.

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eISSN: 2672-815X

Case 5: Numbered t is even. Then one of the numbers x, y, z is even and we reach Case 2 again.

Therefore, any solution is of the form (5.1.2). Of course, c does not divide by b, because otherwise equality (5.1.1′) does not exist.

We observe that any solution of the form (5.1.2) of equation (B), with x and y even numbers it is obtained exactly once by the above formulas. Indeed, from what has been proved above, we obtain that:

$$a=\frac{x}{2}, \qquad b=\frac{y}{2} \qquad \text{and} \qquad c=\frac{t-z}{2};$$

so the integers a, b and c are uniquely determined by the quadruple (x, y, z, t).

Reciprocally, the following identity:

$$(2\cdot a)^2+(2\cdot b)^2+\left(\frac{a^2+b^2-c^2}{c}\right)^2=\left(\frac{a^2+b^2+c^2}{c}\right)^2 \tag{5.1.12}$$

shows that the quadruple defined in the above theorem statement is a solution of equation (B) and, in addition, the numbers x, y are even.

The above theorem not only states the existence of solutions of the given equation, but also provides a practical method for determining all these solutions. It is observed that in order to obtain the solutions of the equation, abstracting from its symmetry in x, y, z it is sufficient to consider only the pairs (a, b), with a≤b and to take only those c for which z is odd. Thus we eliminate the solutions for which x, y, z, t are all even numbers (Cucurezeanu, 2005).

The table below contains the first ten solutions of equation (B) obtained in this way:

a	**b**	**a^2+b^2**	**c**	**x**	**y**	**z**	**t**
1	1	2	1	2	2	1	3
2	2	8	1	4	4	7	9
1	3	10	1	2	6	9	11
1	3	10	2	2	6	3	7
3	3	18	1	6	6	17	19
3	3	18	2	6	6	7	11
3	3	18	3	6	6	3	9
2	4	20	1	4	8	19	21
2	4	20	4	4	8	1	9
4	4	32	1	8	8	31	33

Two remarks are required here:

Remark 5.1.6: *The integer solutions of equation (B) can be expressed in the form*:

$$x=2\cdot a\cdot c, \qquad y=2\cdot b\cdot c, \qquad z=a^2+b^2-c^2 \qquad \text{and} \qquad t=a^2+b^2+c^2, \tag{5.1.13}$$

where a, b, $c\in\mathbf{Z}$. □

We notice that in this form - (5.1.13) - it is possible to obtain, for equation (B), this solution several times. On the other hand, this writing has the advantage that it is very similar to that given for the solutions of equation (A) and is easier to remember.

Remark 5.1.7: *If $(x,y,z,t)\in\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$, is the solution of equation (B), then*:

$$a=y+z-t, \qquad b=x+z-t, \qquad c=x+y-t \qquad \text{and} \qquad d=x+y+z-2\cdot t, \tag{***}$$

it is also a solution of the same equation.

***Proof*:** A simple calculation shows us that:

$a^2=y^2+z^2+t^2+2\cdot y\cdot z-2\cdot y\cdot t-2\cdot z\cdot t,$

$b^2=x^2+z^2+t^2+2\cdot x\cdot z-2\cdot x\cdot t-2\cdot z\cdot t,$

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eISSN: 2672-815X

$c^2=x^2+y^2+t^2+2\cdot x\cdot y-2\cdot x\cdot t-2\cdot y\cdot t,$

$d^2=x^2+y^2+z^2+4\cdot t^2+2\cdot x\cdot y+2\cdot x\cdot z-4\cdot x\cdot t+2\cdot y\cdot z-4\cdot y\cdot t-4\cdot z\cdot t,$

and:

$$\begin{aligned}a^2+b^2+c^2&=2\cdot x^2+2\cdot y^2+2\cdot z^2+3\cdot t^2+2\cdot x\cdot y+2\cdot x\cdot z+2\cdot y\cdot z-4\cdot x\cdot t-4\cdot y\cdot t-4\cdot z\cdot t\\&=2\cdot x^2+2\cdot y^2+2\cdot z^2+3\cdot t^2+2\cdot x\cdot y+2\cdot x\cdot z+2\cdot y\cdot z-4\cdot x\cdot t-4\cdot y\cdot t-4\cdot z\cdot t\\&=x^2+y^2+z^2+4\cdot t^2+2\cdot x\cdot y+2\cdot x\cdot z+2\cdot y\cdot z-4\cdot x\cdot t-4\cdot y\cdot t-4\cdot z\cdot t+x^2+y^2+z^2-t^2,\\&=d^2.\end{aligned}$$

Let us now consider equation (B) in the ring $(M_n(\mathbf{Z}),+,\cdot)$:

$$X^2+Y^2+Z^2=T^2. \tag{B′}$$

The equalities (5.1.13) entitle us to present the following result:

Theorem 5.1.8: *Equation (B′) is solvable in the ring $(M_n(\mathbf{Z}),+,\cdot)$.*

Proof: Indeed, we notice that if A, B, $C\in M_n(\mathbf{Z})$ and verifies the equalities:

$$A\cdot B=B\cdot A,\qquad B\cdot C=C\cdot B,\qquad A\cdot C=C\cdot A, \tag{5.1.14}$$

then the matrices:

$$X=2\cdot A\cdot C,\qquad Y=2\cdot B\cdot C,\qquad Z=A^2+B^2-C^2\quad\text{and}\quad T=A^2+B^2+C^2, \tag{5.1.13′}$$

verifies equality (B′).

Examples 5.1.9: **1)** Starting from Example 3.10.1 of (Vălcan, 2019) we can consider the matrices:

$$A=\begin{pmatrix}4&1\\3&4\end{pmatrix},\qquad B=\begin{pmatrix}2&5\\15&2\end{pmatrix}\qquad\text{and}\qquad C=\begin{pmatrix}5&2\\6&5\end{pmatrix}.$$

Then:

$$A^2=\begin{pmatrix}19&8\\24&19\end{pmatrix},\qquad B^2=\begin{pmatrix}79&20\\60&79\end{pmatrix}\qquad\text{and}\qquad C^2=\begin{pmatrix}37&20\\60&37\end{pmatrix},$$

$$A\cdot B=\begin{pmatrix}23&22\\66&23\end{pmatrix}=B\cdot A,\qquad A\cdot C=\begin{pmatrix}26&13\\39&26\end{pmatrix}=C\cdot A\qquad\text{and}\qquad B\cdot C=\begin{pmatrix}40&29\\87&40\end{pmatrix}=C\cdot B;$$

Now, according to the equalities (5.1.13′) we have:

$$X=\begin{pmatrix}52&26\\78&52\end{pmatrix},\qquad Y=\begin{pmatrix}80&58\\174&80\end{pmatrix},\qquad Z=\begin{pmatrix}61&8\\24&61\end{pmatrix},\qquad T=\begin{pmatrix}135&48\\144&135\end{pmatrix},$$

$$X^2=\begin{pmatrix}4732&2704\\8112&4732\end{pmatrix},\qquad Y^2=\begin{pmatrix}16492&9280\\27840&16492\end{pmatrix},$$

$$Z^2=\begin{pmatrix}3913&976\\2928&3913\end{pmatrix},\qquad T^2=\begin{pmatrix}25137&12960\\38880&25137\end{pmatrix},$$

and the equality (B′) is immediately verified.

2) Let us now consider another example of the solution of the equation (B′), of the form (5.1.13′), with the matrices A, B, $C\in M_2(\mathbf{Z})$; thus be:

$$A=\begin{pmatrix}4&5\\-3&1\end{pmatrix},\qquad B=\begin{pmatrix}3&5\\-3&0\end{pmatrix}\qquad\text{and}\qquad C=\begin{pmatrix}2&5\\-3&-1\end{pmatrix}.$$

Then:

$$A^2=\begin{pmatrix}1&25\\-15&-14\end{pmatrix},\qquad B^2=\begin{pmatrix}-6&15\\-9&-15\end{pmatrix}\qquad\text{and}\qquad C^2=\begin{pmatrix}-11&5\\-3&-14\end{pmatrix},$$

$$A\cdot B=\begin{pmatrix}-3&20\\-12&-15\end{pmatrix}=B\cdot A,\qquad A\cdot C=\begin{pmatrix}-7&15\\-9&-16\end{pmatrix}=C\cdot A\qquad\text{and}\qquad B\cdot C=\begin{pmatrix}-9&10\\-6&-15\end{pmatrix}=C\cdot B.$$

Now, according to the equalities (5.1.13′), we have:

https://doi.org/10.15405/epes.22032.63
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Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2672-815X

$X=\begin{pmatrix} -14 & 30 \\ -18 & -32 \end{pmatrix}$, $Y=\begin{pmatrix} -18 & 20 \\ -12 & -30 \end{pmatrix}$, $Z=\begin{pmatrix} 6 & 35 \\ -21 & -15 \end{pmatrix}$, $T=\begin{pmatrix} -16 & 45 \\ -27 & -43 \end{pmatrix}$,

$X^2=\begin{pmatrix} -344 & -1380 \\ 828 & 484 \end{pmatrix}$, $Y^2=\begin{pmatrix} 84 & -960 \\ 576 & 660 \end{pmatrix}$,

$Z^2=\begin{pmatrix} -699 & -315 \\ 189 & -510 \end{pmatrix}$, $T^2=\begin{pmatrix} -959 & -2655 \\ 1593 & 634 \end{pmatrix}$,

and the equality (B′) is immediately verified.

We can generalize the equations (A), respectively (B) to a certain number n≥2 of variables / unknowns, as follows:

$$x_1^2+x_2^2+\cdots+x_k^2=x_{k+1}^2. \tag{B″}$$

From geometrically if $(x_1,x_2,\ldots,x_k,x_{k+1})\in\underbrace{N^*\times N^*\times\cdots\times N^*}_{k+1}$ is a solution of the equation (B″), then the numbers x_1, x_2, …, x_k represent the dimensions of a right hyperparallelipiped in the space $\mathbf{R}^k$, and the number x_{k+1} is the length of its diagonal. (Andreescu & Andrica, 2002)

Following the same reasoning as in the proof of Theorem 5.1.5, the following theorem can be proved:

Theorem 5.1.10**:** *All integer solutions of the equation (B″) are given by the equations*:

$$x_1=2\cdot a_1\cdot a_k,\quad x_2=2\cdot a_2\cdot a_k,\quad \ldots\quad x_{n-1}=2\cdot a_{k-1}\cdot a_k,$$
$$x_k=a_1^2+a_2^2+\cdots+a_{k-1}^2-a_k^2,\qquad x_{k+1}=a_1^2+a_2^2+\cdots+a_{k-1}^2+a_k^2, \tag{5.1.15}$$

where a_1, a_2, ..., a_{k+1} are any integers.

In the ring $(M_n(\mathbf{Z}),+,\cdot)$ the equation (B″) becomes:

$$X_1^2+X_2^2+\cdots+X_k^2=X_{k+1}^2. \tag{B‴}$$

Equation (B″) is solvable in this ring; moreover, we have the following result:

Theorem 5.1.11**:** *For any matrices A_1, A_2, ..., $A_k\in M_n(\mathbf{Z})$, such that, for every i, j∈{1,2,...,k},*

$$A_i\cdot A_j=A_j\cdot A_i, \tag{5.1.14′}$$

the following system of k+1 matrices in $M_n(\mathbf{Z})$:

$$X_1=2\cdot A_1\cdot A_k,\quad X_2=2\cdot A_2\cdot A_k,\quad \ldots\quad X_{k-1}=2\cdot A_{k-1}\cdot A_k,$$
$$X_k=A_1^2+A_2^2+\cdots+A_{k-1}^2-A_k^2,\qquad X_{k+1}=A_1^2+A_2^2+\cdots+A_{k-1}^2+A_k^2, \tag{5.1.15′}$$

is a solution of the equation (B‴).

Examples 5.1.12**: 1)** Let be $k\in\mathbf{N}^*$, k≥2. Starting from Example 3.7.1 of (Vălcan, 2019), for every $i\in\{1,2,...,k\}$, we can consider the matrices:

$$A_i=\begin{pmatrix} a_i & 0 \\ c_i & a_i \end{pmatrix}.$$

Then, for every i, j∈{1,2,…,k}:

$$A_i^2=\begin{pmatrix} a_i^2 & 0 \\ 2\cdot a_i\cdot c_i & a_i^2 \end{pmatrix},\qquad \text{and}\qquad A_i\cdot A_j=\begin{pmatrix} a_i\cdot a_j & 0 \\ a_i\cdot c_j+c_i\cdot a_j & a_i\cdot a_j \end{pmatrix}=A_j\cdot A_i,$$

Now, according to the equalities (5.1.15′), for every $i=\overline{1,k-1}$, we have:

$$X_i=\begin{pmatrix} 2\cdot a_i\cdot a_k & 0 \\ 2\cdot(a_i\cdot c_k+c_i\cdot a_k) & 2\cdot a_i\cdot a_k \end{pmatrix},$$

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eISSN: 2672-815X

$$X_i^2=\begin{pmatrix} 4\cdot a_i^2\cdot a_k^2 & 0 \\ 8\cdot a_i\cdot a_k\cdot(a_i\cdot c_k+c_i\cdot a_k) & 4\cdot a_i^2\cdot a_k^2 \end{pmatrix},$$

$$X_k=\begin{pmatrix} a_1^2+a_2^2+\cdots+a_{k-1}^2-a_k^2 & 0 \\ 2\cdot(a_1\cdot c_1+a_2\cdot c_2+\cdots+a_{k-1}\cdot c_{k-1}-a_k\cdot c_k) & a_1^2+a_2^2+\cdots+a_{k-1}^2-a_k^2 \end{pmatrix}$$

$$X_{k+1}=\begin{pmatrix} a_1^2+a_2^2+\cdots+a_{k-1}^2+a_k^2 & 0 \\ 2\cdot(a_1\cdot c_1+a_2\cdot c_2+\cdots+a_{k-1}\cdot c_{k-1}+a_k\cdot c_k) & a_1^2+a_2^2+\cdots+a_{k-1}^2+a_k^2 \end{pmatrix},$$

$$X_k^2=\begin{pmatrix} \alpha_k & 0 \\ \beta_k & \alpha_k \end{pmatrix} \quad \text{and} \quad X_{k+1}^2=\begin{pmatrix} \gamma_k & 0 \\ \delta_k & \gamma_k \end{pmatrix},$$

where:

$\alpha_k=(a_1^2+a_2^2+\cdots+a_{k-1}^2-a_k^2)^2$,

$\beta_k=4\cdot(a_1^2+a_2^2+\cdots+a_{k-1}^2-a_k^2)\cdot(a_1\cdot c_1+a_2\cdot c_2+\cdots a_{k-1}\cdot c_{k-1}-a_k\cdot c_k)$,

$\gamma_k=(a_1^2+a_2^2+\cdots+a_{k-1}^2+a_k^2)^2$,

$\beta_k=4\cdot(a_1^2+a_2^2+\cdots+a_{k-1}^2+a_k^2)\cdot(a_1\cdot c_1+a_2\cdot c_2+\cdots a_{k-1}\cdot c_{k-1}+a_k\cdot c_k)$,

and the equality (B‴) is immediately verified.

2) Fie $k\in\mathbf{N}^*$, $k\geq 2$. Starting from Example 3.8.1 of (Vălcan, 2019), for any $i\in\{1,2,...,k\}$, we can consider the matrices:

$$A_i=\begin{pmatrix} a_i & b_i \\ 0 & a_i \end{pmatrix}.$$

Then, for every $i, j\in\{1,2,\dots,k\}$:

$$A_i^2=\begin{pmatrix} a_i^2 & 2\cdot a_i\cdot b_i \\ 0 & a_i^2 \end{pmatrix}, \quad \text{and} \quad A_i\cdot A_j=\begin{pmatrix} a_i\cdot a_j & a_i\cdot c_j+c_i\cdot a_j 0 \\ 0 & a_i\cdot a_j \end{pmatrix}=A_j\cdot A_i,$$

Now, according to the equalities (5.1.15′), for every $i=\overline{1,k-1}$, we have:

$$X_i=\begin{pmatrix} 2\cdot a_i\cdot a_k & 2\cdot(a_i\cdot b_k+c_i\cdot b_k) \\ 0 & 2\cdot a_i\cdot a_k \end{pmatrix},$$

$$X_i^2=\begin{pmatrix} 4\cdot a_i^2\cdot a_k^2 & 8\cdot a_i\cdot a_k\cdot(a_i\cdot b_k+b_i\cdot a_k) \\ 0 & 4\cdot a_i^2\cdot a_k^2 \end{pmatrix},$$

$$X_k=\begin{pmatrix} a_1^2+a_2^2+\cdots+a_{k-1}^2-a_k^2 & 2\cdot(a_1\cdot b_1+a_2\cdot b_2+\cdots+a_{k-1}\cdot b_{k-1}-a_k\cdot b_k) \\ 0 & a_1^2+a_2^2+\cdots+a_{k-1}^2-a_k^2 \end{pmatrix}$$

$$X_{k+1}=\begin{pmatrix} a_1^2+a_2^2+\cdots+a_{k-1}^2+a_k^2 & 2\cdot(a_1\cdot b_1+a_2\cdot b_2+\cdots+a_{k-1}\cdot b_{k-1}+a_k\cdot b_k) \\ 0 & a_1^2+a_2^2+\cdots+a_{k-1}^2+a_k^2 \end{pmatrix},$$

$$X_k^2=\begin{pmatrix} \varepsilon_k & \phi_k \\ 0 & \varepsilon_k \end{pmatrix} \quad \text{and} \quad X_{k+1}^2=\begin{pmatrix} \varphi_k & \eta_k \\ 0 & \varphi_k \end{pmatrix},$$

where:

$\varepsilon_k=(a_1^2+a_2^2+\cdots+a_{k-1}^2-a_k^2)^2$,

$\phi_k=4\cdot(a_1^2+a_2^2+\cdots+a_{k-1}^2-a_k^2)\cdot(a_1\cdot b_1+a_2\cdot b_2+\cdots a_{k-1}\cdot b_{k-1}-a_k\cdot b_k)$,

$\varphi_k=(a_1^2+a_2^2+\cdots+a_{k-1}^2+a_k^2)^2$,

$\eta_k=4\cdot(a_1^2+a_2^2+\cdots+a_{k-1}^2+a_k^2)\cdot(a_1\cdot b_1+a_2\cdot b_2+\cdots a_{k-1}\cdot b_{k-1}+a_k\cdot b_k)$,

https://doi.org/10.15405/epes.22032.63
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Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2672-815X

and the equality (B‴) is immediately verified.

5.2. Diofantiene quadratic equations

We begin this paragraph by examining the Diophantine equation:

$$x^2+a\cdot x\cdot y+b\cdot y^2=z^2. \tag{C}$$

where a and b are given integers. The Pythagorean equation (A) is a special case of this equation which is obtained for:

$$a=0 \quad \text{and} \quad b=1.$$

Regarding equation (C) we have the following result:

Theorem 5.2.1: *All integer solutions of equation (C) are given by the equalities*:

$$x=k\cdot(m^2\cdot b-n^2),\qquad y=k\cdot(2\cdot m\cdot n-m^2\cdot a)\qquad \text{and}\qquad z=k\cdot(m^2\cdot b-m\cdot n\cdot a+n^2), \tag{5.2.1}$$

where k, m, n $\in\mathbf{Z}$.

***Proof*:** A simple calculation shows that the triplets (5.2.1) - defined in the theorem statement, satisfy equation (C). Indeed,

$$x^2=k^2\cdot(m^4\cdot b^2+n^4-2\cdot m^2\cdot n^2\cdot b), \tag{5.2.2}$$

$$a\cdot x\cdot y=k^2\cdot(2\cdot m^3\cdot n\cdot a\cdot b-m^4a^2\cdot b-2\cdot m\cdot n^3\cdot a+m^2\cdot n^2\cdot a^2), \tag{5.2.3}$$

$$b\cdot y^2=k^2\cdot(4\cdot m^2\cdot n^2\cdot b-4\cdot m^3\cdot n\cdot a\cdot b+m^4\cdot a^2\cdot b). \tag{5.2.4}$$

Now, from the equalities (5.2.2), (5.2.3) and (5.2.4), we obtain:

$$x^2+a\cdot x\cdot y+b\cdot y^2=k^2\cdot(m^4\cdot b^2+m^2\cdot n^2\cdot a^2+n^4-2\cdot m^3\cdot n\cdot a\cdot b+2\cdot m^2\cdot n^2\cdot b-2\cdot m\cdot n^3\cdot a)$$
$$=z^2.$$

Conversely, we will show that all solutions of the equation are of the form above. For this, we observe that equation (C) is equivalent to:

$$y\cdot(a\cdot x+b\cdot y)=(z-x)\cdot(z+x). \tag{C′}$$

If

$$x=z,$$

then:

$$y=0 \quad \text{or} \quad a\cdot x+b\cdot y=0. \tag{5.2.5}$$

In these situations, all triplets of the form (x,0,x) and, respectively $\left(x,-\frac{a\cdot x}{b},x\right)$, with a, b, x $\in\mathbf{Z}$ and b | (a·x) are solutions of equation (C). The first sub-case corresponds to the situation when:

$$m\cdot a=2\cdot n,$$

and the second sub-case occurs when:

$$n=0.$$

In all other cases the equation (C′) is equivalent to:

$$\frac{y}{z-x}=\frac{z+x}{a\cdot x+b\cdot y}\overset{\text{not.}}{=}\frac{m}{n}, \tag{C″}$$

where m, n $\in\mathbf{Z}$. These equalities lead us to the homogeneous system:

$$\begin{cases} m\cdot x+n\cdot y-m\cdot z=0 \\ (m\cdot a-n)\cdot x+m\cdot y-n\cdot z=0 \end{cases}, \tag{5.2.6}$$

whose solutions are:

$$x=\frac{m^2\cdot b-n^2}{m^2\cdot b-a\cdot m\cdot n+n^2}\cdot z \quad \text{and} \quad y=\frac{2\cdot m\cdot n-m^2\cdot a}{m^2\cdot b-a\cdot m\cdot n+n^2}\cdot z. \tag{5.2.1′}$$

Now, we choose:

$$z=k\cdot(m^2\cdot b-a\cdot m\cdot n+n^2), \tag{5.2.1″}$$

with $k\in\mathbf{Z}$ and obtain the solutions (5.2.1).

In the ring $(M_n(\mathbf{Z}),+,\cdot)$ the equation (C) can be of the form:

$$X^2+a\cdot X\cdot Y+b\cdot Y^2=Z^2, \tag{C‴}$$

where a and b are given integers, or may be of the form:

$$X^2+A\cdot X\cdot Y+B\cdot Y^2=Z^2, \tag{C$^{(iv)}$}$$

where $A, B\in M_n(\mathbf{Z})$ are fixed.

The equalities (5.2.1) entitle us to present the following result:

Theorem 5.2.2: *Equations (C‴) and (C$^{(iv)}$), respectively) are solvable in the ring $(M_n(\mathbf{Z}),+,\cdot)$.*

Proof: Indeed, we notice that if $a, b\in\mathbf{Z}$, and $A, B, M, N\in M_n(\mathbf{Z})$ and satisfy equalities:

$$A\cdot B=B\cdot A, \quad A\cdot M=M\cdot A, \quad M\cdot N=N\cdot M,$$
$$A\cdot N=N\cdot A, \quad B\cdot M=M\cdot B, \quad B\cdot N=N\cdot B, \tag{5.2.7}$$

then the matrices:

$$X=b\cdot M^2-N^2, \quad Y=2\cdot M\cdot N-a\cdot M^2 \quad \text{and} \quad Z=b\cdot M^2-a\cdot M\cdot N+N^2 \tag{5.2.8}$$

verifies the equality (C‴), and the matrices:

$$X_1=B\cdot M^2-N^2, \quad Y_1=2\cdot M\cdot N-A\cdot M^2 \quad \text{and} \quad Z_1=B\cdot M^2-A\cdot M\cdot N+N^2 \tag{5.2.9}$$

verifies the equality (C$^{(iv)}$).

Examples 5.2.3: Starting also from Example 3.10.1 from (Vălcan, 2019) we can consider the matrices:

$$A=\begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}, \quad B=\begin{pmatrix}3 & 5\\ 15 & 3\end{pmatrix}, \quad M=\begin{pmatrix}4 & 15\\ 45 & 4\end{pmatrix} \quad \text{and} \quad N=\begin{pmatrix}5 & 2\\ 6 & 5\end{pmatrix}.$$

Then:

$$M^2=\begin{pmatrix}691 & 120\\ 360 & 691\end{pmatrix} \quad \text{and} \quad N^2=\begin{pmatrix}37 & 20\\ 60 & 37\end{pmatrix},$$

$$A\cdot B=\begin{pmatrix}21 & 13\\ 39 & 21\end{pmatrix}=B\cdot A, \quad A\cdot M=\begin{pmatrix}53 & 34\\ 102 & 53\end{pmatrix}=M\cdot A, \quad M\cdot N=\begin{pmatrix}110 & 83\\ 249 & 110\end{pmatrix}=N\cdot M,$$

$$A\cdot N=\begin{pmatrix}16 & 9\\ 27 & 16\end{pmatrix}=N\cdot A, \quad B\cdot M=\begin{pmatrix}237 & 65\\ 195 & 237\end{pmatrix}=M\cdot A, \quad B\cdot N=\begin{pmatrix}45 & 31\\ 93 & 45\end{pmatrix}=B\cdot N.$$

Now, according to equality (5.2.8), for:

$$b=3 \quad \text{and} \quad a=2,$$

we have:

$$X=\begin{pmatrix}2036 & 340\\ 1020 & 2036\end{pmatrix}, \quad Y=\begin{pmatrix}-1162 & -74\\ -222 & -1162\end{pmatrix} \quad \text{and} \quad Z=\begin{pmatrix}1890 & 214\\ 642 & 1890\end{pmatrix},$$

$$X^2=\begin{pmatrix}4492096 & 1384480\\ 4153440 & 4492096\end{pmatrix}, \quad X\cdot Y=\begin{pmatrix}-2441312 & -545744\\ -1637232 & -2441312\end{pmatrix}=Y\cdot X,$$

$$Y^2=\begin{pmatrix}1366672 & 171976\\ 515928 & 1366672\end{pmatrix}, \quad Z^2=\begin{pmatrix}3709488 & 808920\\ 2426760 & 3709488\end{pmatrix},$$

and the equality (C‴), for the case considered, is verified immediately:

$$X^2+2\cdot X\cdot Y+3\cdot Y^2=\begin{pmatrix}4492096 & 1384480\\ 4153440 & 4492096\end{pmatrix}+\begin{pmatrix}-4882624 & -1091488\\ -3274464 & -4882624\end{pmatrix}+\begin{pmatrix}4100016 & 515928\\ 1547784 & 4100016\end{pmatrix}$$
$$=\begin{pmatrix}3709488 & 808920\\ 2426760 & 3709488\end{pmatrix}$$
$$=Z^2.$$

On the other hand, according to the equalities (5.2.9), we obtain:

https://doi.org/10.15405/epes.22032.63
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Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2672-815X

$X_1=\begin{pmatrix} 3836 & 3795 \\ 11385 & 3836 \end{pmatrix}$, $Y_1=\begin{pmatrix} -1522 & -765 \\ -2295 & -1522 \end{pmatrix}$ and $Z_1=\begin{pmatrix} 3441 & 3559 \\ 10677 & 3441 \end{pmatrix}$,

$X_1^2=\begin{pmatrix} 57920971 & 29115240 \\ 87345720 & 57920971 \end{pmatrix}$, $X_1\cdot Y_1=\begin{pmatrix} -14547917 & -8710530 \\ -26131590 & -14547917 \end{pmatrix}=Y_1\cdot X_1$,

$Y_1^2=\begin{pmatrix} 4072159 & 2328660 \\ 6985980 & 4072159 \end{pmatrix}$, $Z_1^2=\begin{pmatrix} 49839924 & 24493038 \\ 73479114 & 49839924 \end{pmatrix}$,

and the equality ($C^{(iv)}$), for the case considered, is verified immediately:

$$X_1^2+A\cdot X_1\cdot Y_1+B\cdot Y_1^2=\begin{pmatrix} 57920971 & 29115240 \\ 87345720 & 57920971 \end{pmatrix}-\begin{pmatrix} 55227424 & 31968977 \\ 95906931 & 55227424 \end{pmatrix}$$
$$+\begin{pmatrix} 47146377 & 27346775 \\ 82040325 & 47146377 \end{pmatrix}=\begin{pmatrix} 49839924 & 24493038 \\ 73479114 & 49839924 \end{pmatrix}$$
$$=Z_1^2 .$$

A particular case of equation (C) we obtain for:

$b=a^2$,

when equation (C) becomes:

$$x^2+a\cdot x\cdot y+a^2\cdot y^2=z^2. \qquad (D)$$

From Theorem 5.2.1 we obtain immediately:

Corollary 5.2.4: *The integer solutions of equation (D) are*:

$$x=k\cdot(m^2\cdot a^2-n^2),\quad y=k\cdot(2\cdot m\cdot n-m^2\cdot a)\quad \text{and}\quad z=k\cdot(m^2\cdot a^2-m\cdot n\cdot a+n^2), \qquad (5.2.10)$$

where k, m, $n\in\boldsymbol{Z}$.

In the ring $(M_n(\boldsymbol{Z}),+,\cdot)$ the equation (D) can be of the form:

$$X^2+a\cdot X\cdot Y+a^2\cdot Y^2=Z^2, \qquad (D')$$

where a is an integer, given, or can be of the form:

$$X^2+A\cdot X\cdot Y+A^2\cdot Y^2=Z^2, \qquad (D'')$$

where $A\in M_n(\boldsymbol{Z})$ is fixed.

The equalities (5.2.10) entitle us to present the following result:

Theorem 5.2.5: *Equations (D′) and (D″), respectively, are solvable in the ring* $(M_n(\boldsymbol{Z}),+,\cdot)$.

Proof: Indeed, we observe that if $\in\boldsymbol{Z}$, and A, M, $N\in M_n(\boldsymbol{Z})$ and satisfy the equalities:

$$A\cdot M=M\cdot A,\qquad M\cdot N=N\cdot M,\qquad A\cdot N=N\cdot A, \qquad (5.2.11)$$

then the matrices:

$$X=a^2\cdot M^2-N^2,\qquad Y=2\cdot M\cdot N-a\cdot M^2\quad \text{and}\quad Z=a^2\cdot M^2-a\cdot M\cdot N+N^2 \qquad (5.2.12)$$

verifies the equality (D′) and the matrices:

$$X_1=A^2\cdot M^2-N^2,\qquad Y_1=2\cdot M\cdot N-A\cdot M^2\quad \text{and}\quad Z_1=A^2\cdot M^2-A\cdot M\cdot N+N^2 \qquad (5.2.13)$$

verifies the equality (D″).

Examples 5.2.6: We also consider the matrices here:

$A=\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, $M=\begin{pmatrix} 4 & 15 \\ 45 & 4 \end{pmatrix}$, and $N=\begin{pmatrix} 5 & 2 \\ 6 & 5 \end{pmatrix}$.

Then:

$A^2=\begin{pmatrix} 7 & 4 \\ 12 & 7 \end{pmatrix}$, $M^2=\begin{pmatrix} 691 & 120 \\ 360 & 691 \end{pmatrix}$ and $N^2=\begin{pmatrix} 37 & 20 \\ 60 & 37 \end{pmatrix}$,

$A\cdot M=\begin{pmatrix} 53 & 34 \\ 102 & 53 \end{pmatrix}=M\cdot A$, $M\cdot N=\begin{pmatrix} 110 & 83 \\ 249 & 110 \end{pmatrix}=N\cdot M$, $A\cdot N=\begin{pmatrix} 16 & 9 \\ 27 & 16 \end{pmatrix}=N\cdot A$.

Now, according to the equalities (5.2.12), for:

https://doi.org/10.15405/epes.22032.63
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Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2672-815X

a=2,

we have:

$$X=\begin{pmatrix}2727 & 460\\ 1380 & 2727\end{pmatrix},\qquad Y=\begin{pmatrix}-1162 & -74\\ -222 & -1162\end{pmatrix},\qquad \text{and}\qquad Z=\begin{pmatrix}2581 & 334\\ 1002 & 2581\end{pmatrix},$$

$$X^2=\begin{pmatrix}8071329 & 2508840\\ 7526520 & 8071329\end{pmatrix},\qquad X\cdot Y=\begin{pmatrix}-3270894 & -736318\\ -2208954 & -3270894\end{pmatrix}=Y\cdot X,$$

$$Y^2=\begin{pmatrix}1366672 & 171976\\ 515928 & 1366672\end{pmatrix},\qquad Z^2=\begin{pmatrix}6996229 & 1724108\\ 5172324 & 6996229\end{pmatrix},$$

and equality (D′), for the case in question, is verified immediately:

$$X^2+2\cdot X\cdot Y+4\cdot Y^2=\begin{pmatrix}8071329 & 2508840\\ 7526520 & 8071329\end{pmatrix}-\begin{pmatrix}6541788 & 1472636\\ 4417908 & 6541788\end{pmatrix}+\begin{pmatrix}5466688 & 687904\\ 2063712 & 5466688\end{pmatrix}$$

$$=\begin{pmatrix}6996229 & 1724108\\ 5172324 & 696229\end{pmatrix}$$

$$=Z^2.$$

On the other hand, according to the equalities (5.2.13), we obtain:

$$X_1=\begin{pmatrix}6240 & 3584\\ 10752 & 6240\end{pmatrix},\qquad Y_1=\begin{pmatrix}-1522 & -765\\ -2295 & -1522\end{pmatrix},\qquad \text{and}\qquad Z_1=\begin{pmatrix}5845 & 3348\\ 10044 & 5845\end{pmatrix},$$

$$X_1^2=\begin{pmatrix}77472768 & 44728320\\ 134184960 & 77472768\end{pmatrix},\qquad X_1\cdot Y_1=\begin{pmatrix}-17722560 & -10228448\\ -30685344 & -17722560\end{pmatrix}=Y_1\cdot X_1,$$

$$Y_1^2=\begin{pmatrix}4072159 & 2328660\\ 6985980 & 4072159\end{pmatrix},\qquad Z_1^2=\begin{pmatrix}67791337 & 39138120\\ 117414360 & 67791337\end{pmatrix},$$

and the equality (D″), for the case considered, is verified immediately:

$$X_1^2+A\cdot X_1\cdot Y_1+A^2\cdot Y_1^2=\begin{pmatrix}77472768 & 44728320\\ 134184960 & 77472768\end{pmatrix}-\begin{pmatrix}66130464 & 38179456\\ 114538368 & 66130464\end{pmatrix}$$

$$+\begin{pmatrix}56449033 & 32589256\\ 97767768 & 56449033\end{pmatrix}=\begin{pmatrix}67791337 & 39138120\\ 117414360 & 67791337\end{pmatrix}$$

$$=Z_1^2.$$

The following two particular cases of equation (D) are interesting:

1) a=1. In this case equation (D) becomes:

$$x^2+x\cdot y+y^2=z^2. \tag{D‴}$$

From the equalities (5.2.10) it results that its non-zero natural solutions are given by:

$$x=k\cdot(m^2-n^2),\qquad y=k\cdot(2\cdot m\cdot n-m^2)\qquad \text{and}\qquad z=k\cdot(m^2-m\cdot n+n^2), \tag{5.2.14}$$

where k, m, $n\in\mathbf{N}^*$, and $m>n>\frac{m}{2}$.

The solutions (5.2.14) give all triplets (x, y, z) of nonzero natural numbers which are the lengths of the sides of a triangle with the angle opposite to the side of length z equal to 120°. (Andreescu & Andrica, 2002)

2) a=-1. In this case equation (D) becomes:

$$x^2-x\cdot y+y^2=z^2. \tag{D$^{(iv)}$}$$

From the same equalities (5.2.10) it results that its non-zero natural solutions are given by the equalities:

$$x=k\cdot(m^2-n^2),\qquad y=k\cdot(2\cdot m\cdot n+m^2)\qquad \text{and}\qquad z=k\cdot(m^2+m\cdot n+n^2), \tag{5.2.15}$$

https://doi.org/10.15405/epes.22032.63
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Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2672-815X

where k, m, n$\in$**N***, and m>n.

The solutions (5.2.15) characterize all triplets of non-zero natural numbers (x, y, z) which are the lengths of the sides of a triangle having the angle opposite to the side of length z equal to 60° (Andreescu & Andrica, 2002)

In the ring $(M_n(\mathbf{Z}),+,\cdot)$ the equations (D''') and $(D^{(iv)})$ respectively become:

$$X^2+X\cdot Y+Y^2=Z^2 \qquad (D^{(v)})$$

respectively:

$$X^2-X\cdot Y+Y^2=Z^2. \qquad (D^{(vi)})$$

The results obtained above entitle us to present:

Corollary 5.2.7: *Equations $(D^{(v)})$ and $(D^{(vi)})$, respectively, are solvable in the ring $(M_n(\mathbf{Z}),+,\cdot)$.*

Proof: Indeed, we observe that if M, N$\in M_n(\mathbf{Z})$ and satisfy the equality:

$$M\cdot N=N\cdot M, \qquad (5.2.16)$$

then the matrices:

$$X=M^2-N^2, \qquad Y=2\cdot M\cdot N-M^2 \qquad \text{and} \qquad Z=M^2-M\cdot N+N^2 \qquad (5.2.17)$$

verifies the equality $(D^{(v)})$, and the matrices:

$$X_1=M^2-N^2, \qquad Y_1=2\cdot M\cdot N+M^2 \qquad \text{and} \qquad Z_1=M^2+M\cdot N+N^2 \qquad (5.2.18)$$

verifies the equality $(D^{(vi)})$.

***Examples 5.2.8*: 1)** We can consider the matrices in Examples 5.2.3:

$$M=\begin{pmatrix} 4 & 15 \\ 45 & 4 \end{pmatrix}, \qquad \text{and} \qquad N=\begin{pmatrix} 5 & 2 \\ 6 & 5 \end{pmatrix}.$$

Then:

$$M^2=\begin{pmatrix} 691 & 120 \\ 360 & 691 \end{pmatrix}, \qquad N^2=\begin{pmatrix} 37 & 20 \\ 60 & 37 \end{pmatrix} \qquad \text{and} \qquad M\cdot N=\begin{pmatrix} 110 & 83 \\ 249 & 110 \end{pmatrix}=N\cdot M.$$

Now, according to the equalities (5.2.17), we have:

$$X=\begin{pmatrix} 654 & 100 \\ 300 & 654 \end{pmatrix}, \qquad Y=\begin{pmatrix} -471 & 46 \\ 138 & -471 \end{pmatrix}, \qquad \text{and} \qquad Z=\begin{pmatrix} 618 & 57 \\ 171 & 618 \end{pmatrix},$$

$$X^2=\begin{pmatrix} 457716 & 130800 \\ 392400 & 457716 \end{pmatrix}, \qquad X\cdot Y=\begin{pmatrix} -294234 & -17016 \\ -51048 & -294234 \end{pmatrix}=Y\cdot X,$$

$$Y^2=\begin{pmatrix} 228189 & -43332 \\ -129996 & 228189 \end{pmatrix}, \qquad Z^2=\begin{pmatrix} 391671 & 70452 \\ 211356 & 391671 \end{pmatrix},$$

and the equality $(D^{(v)})$, is verified immediately:

$$X^2+X\cdot Y+Y^2=\begin{pmatrix} 457716 & 130800 \\ 392400 & 457716 \end{pmatrix}+\begin{pmatrix} -294234 & -17016 \\ -51048 & -294234 \end{pmatrix}+\begin{pmatrix} 228189 & -43332 \\ -129996 & 228189 \end{pmatrix}$$

$$=\begin{pmatrix} 391671 & 70452 \\ 211356 & 391671 \end{pmatrix}$$

$$=Z^2.$$

On the other hand, according to the equalities (5.2.9), we obtain:

$$X_1=\begin{pmatrix} 654 & 100 \\ 300 & 654 \end{pmatrix}, \qquad Y_1=\begin{pmatrix} 911 & 286 \\ 858 & 911 \end{pmatrix}, \qquad \text{and} \qquad Z_1=\begin{pmatrix} 838 & 223 \\ 669 & 838 \end{pmatrix},$$

$$X_1^2=\begin{pmatrix} 457716 & 130800 \\ 392400 & 457716 \end{pmatrix}, \qquad X_1\cdot Y_1=\begin{pmatrix} 681594 & 278144 \\ 834432 & 681594 \end{pmatrix}=Y_1\cdot X_1,$$

$$Y_1^2=\begin{pmatrix} 1075309 & 521092 \\ 1563276 & 1075309 \end{pmatrix}, \qquad Z_1^2=\begin{pmatrix} 851431 & 373748 \\ 1121244 & 851431 \end{pmatrix},$$

https://doi.org/10.15405/epes.22032.63
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Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2672-815X

and the equality ($C^{(iv)}$), for the case considered, is verified immediately:

$$X_1^2 - X_1 \cdot Y_1 + Y_1^2 = \begin{pmatrix} 457716 & 130800 \\ 392400 & 457716 \end{pmatrix} - \begin{pmatrix} 681594 & 278144 \\ 834432 & 681594 \end{pmatrix} + \begin{pmatrix} 1075309 & 521092 \\ 1563276 & 1075309 \end{pmatrix}$$

$$= \begin{pmatrix} 851431 & 373748 \\ 1121244 & 851431 \end{pmatrix}$$

$$= Z_1^2 .$$

The following remarks are required here:

Remarks 5.2.9: 1) *Theorem 5.2.1 shows how to solve the following third degree Diophantine equation*:

$$x^2+x\cdot y\cdot t+y^2=z^2. \tag{E}$$

Its general solution is (x,y,z,t), where:

$t=a$,

with $a\in\mathbf{Z}$ *and x, y, z are given by the equalities (5.2.1) - for b=1.*

2) *Combining the results proved above we can solve the Diophantine equation*:

$$x^2+u\cdot x\cdot y+v\cdot y^2=z^2. \tag{F}$$

Its solutions are (x,y,z,t,u,v), where:

$u=a$ *and* $v=b$,

with $a, b\in\mathbf{Z}$*, and x, y, z are given by the equalities (5.2.1).*

In the ring $(M_n(\mathbf{Z}),+,\cdot)$ the equations (E) and (F) respectively become:

$$X^2+X\cdot Y\cdot T+Y^2=Z^2 \tag{E′}$$

respectively:

$$X^2+U\cdot X\cdot Y+V\cdot Y^2=Z^2. \tag{F′}$$

And at the end of this paragraph we can present:

Corollary 5.2.10: *Equations (E′) and (F′), respectively, are solvable in the ring* $(M_n(\mathbf{Z}),+,\cdot)$.

Proof: Indeed, we observe that if $A, B, M, N\in M_n(\mathbf{Z})$ and satisfy the equalities:

$A\cdot M=M\cdot A$, $M\cdot N=N\cdot M$, $A\cdot N=N\cdot A$, (5.2.11)

$B\cdot M=M\cdot B$, $M\cdot N=N\cdot M$, $B\cdot N=N\cdot B$, (5.2.11)

and moreover:

$$A\cdot B=B\cdot A, \tag{5.2.16}$$

then, according to the proof of Theorem 5.2.2, the matrices:

$X=M^2-N^2$, $Y=2\cdot M\cdot N-A\cdot M^2$, $Z=M^2-A\cdot M\cdot N+N^2$ and $T=A$ (5.2.13′)

verifies the equality (E′), and the matrices:

$X_1=B\cdot M^2-N^2$, $Y_1=2\cdot M\cdot N-A\cdot M^2$, $Z_1=B\cdot M^2-A\cdot M\cdot N+N^2$,

$U=A$ and $V=B$ (5.2.9)

verifies the equality (F′).

6. Findings

Therefore, not only equations of form (A) but also any equations of form (B), (C), (D), (E) or (F), can be transposed into the ring $(M_2(\mathbf{Z}),+,\cdot)$, where it has solutions. Moreover, each of the solutions determined in Paragraph 5 induces a solution $(X^{(n)},Y^{(n)},Z^{(n)})\in M_n(\mathbf{Z})\times M_n(\mathbf{Z})\times M_n(\mathbf{Z})$, according to the model presented in Paragraph 4 of the paper (Vălcan, 2019).

7. Conclusion

https://doi.org/10.15405/epes.22032.63
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Selection and peer-review under responsibility of the Organizing Committee of the conference
eISSN: 2672-815X

As a general conclusion, we can say that any equation of the form (A), (B), (C), (D), (E) or (F) can be "*immersed*" in a ring of matrices of the type ($M_n(\mathbf{Z}),+,\cdot$), with $n\in\mathbf{N}^*$, any number, at least equal to 2; only that, if these equations can be solved completely in the ring of integers ($\mathbf{Z},+,\cdot$), i.e. all their integer solutions can be determined, the same cannot be said about the corresponding "*submerged*" equation, i.e. in this ring of matrices determining all the solutions is usually quite difficult and then only certain solutions are determined.

Of course, this paper is one of Didactics of Mathematics and is addressed to pupils, students or teachers attentive and interested in these issues, which we believe we have formed, in this way, a good image about solving these two types of equations.

References

Acu, D. (2010). *Aritmetica si teoria numerelor* [Arithmetic and Number Theory]. Editura Universității *"Lucian Blaga"*.

Andreescu, T., & Andrica, D. (2002). *O introducere in studiul ecuațiilor diofantiene* [An Introduction to the Study of Diofantian Equations]. Editura GIL.

Cucurezeanu, I. (2005). *Ecuaţii în numere întregi* [Equations in integer numbers]. Editura Aramis Print.

Vălcan, D. (2017). *FORMAREA INIŢIALĂ A PROFESORILOR DE MATEMATICĂ LA UNIVERSITATEA "BABEŞ-BOLYAI", NIVEL LICENŢĂ, Curriculumul Național matematic preuniversitar în perioada 2000 - 2016*, Vol. I: *De la obiective / competențe la planuri - cadru de învățământ* [INITIAL TRAINING OF MATHEMATICS TEACHERS IN "BABEŞ-BOLYAI" UNIVERSITY, LICENSE LEVEL, National Pre - University Mathematical Curriculum in the Period 2000 - 2016, Vol. I: From Objectives / Competencies to curriculum framework plans], Cluj-Napoca, Editura Casa Cărţii de Ştiinţă.

Vălcan, D. (2019). *From Diofantian Equations to Matricial Equations (I) - Equations and Pythagorean Matrices*, in *Journal of Education & Social Policy*, *6*(1), 60-83.